

# STRANGE NON-CHAOTIC ATTRACTORS IN QUASIPERIODICALLY FORCED CIRCLE MAPS

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## Abstract

The occurrence of strange non-chaotic attractors (SNA) in quasiperiodically forced systems has attracted considerable interest over the last two decades, in particular since it provides a rich class of examples for the possibility of complicated dynamics in the absence of chaos. Their existence was discovered in the early 1980's, independently by Herman [1] for quasiperiodic  $SL(2, \mathbb{R})$ -cocycles and by Grebogi *et al* [2] for so-called '*pinched skew products*'. However, except for these two particular classes there are still hardly any rigorous results on the topic, despite a large number of numerical studies which all confirmed the widespread existence of SNA in quasiperiodically forced systems.

Here, we prove the existence of SNA in quasiperiodically forced circle maps under rather general conditions, which can be stated in terms of  $\mathcal{C}^1$ -estimates. As a consequence, we obtain the existence of strange non-chaotic attractors for parameter sets of positive measure in suitable parameter families. Further, we show that the considered systems have minimal dynamics. The results apply in particular to a forced version of the Arnold circle map. For this particular example, we also describe how the first Arnold tongue collapses and loses its regularity due to the presence of strange non-chaotic attractors and a related unbounded mean motion property.

## 1 Introduction

In 1984, Grebogi *et al* [2] introduced a class of quasiperiodically forced (qpf) interval maps which exhibit non-continuous invariant graphs with negative (vertical) Lyapunov exponents. As these objects attract a set of initial conditions of positive measure and combine a complicated structure with non-chaotic dynamics (in particular zero topological entropy), they are commonly referred to as *strange non-chaotic attractors* (SNA). Already one year earlier, Herman [1] had proved the existence of such SNA in certain parameter families of qpf circle diffeomorphisms that are induced by the projective action of  $SL(2, \mathbb{R})$ -cocycles over an irrational rotation (see also [3]).

In the following years, the phenomenon attracted a considerable amount of interest, and a large number of numerical studies indicated that the existence of SNA is quite common in quasiperiodically forced systems ([4] gives a good overview and further reference). However, despite all efforts rigorous results remained rare, and in particular the two classes of examples mentioned above remained the only ones for which the existence of SNA could be proved rigorously. Only recently some further progress was made, as the author described the creation of SNA in non-smooth bifurcations of invariant curves, which take place in qpf interval maps [5] (but only at isolated parameter values).

The aim of this article is two-fold. First, we show that once the skew-product structure is given, which is usually motivated by the physical context of the model, the existence of SNA in qpf circle maps is a phenomenon which is both 'robust' and 'non-degenerate'. To make this more precise, we denote by  $\text{Diff}_0(\mathbb{T}^2)$  the set of all

diffeomorphisms of the two-torus which are homotopic to the identity and by  $\pi_i$  the projection to the respective coordinate. Further, for any  $\omega \in \mathbb{T}^1$  we let  $R_\omega(\theta, x) = (\theta + \omega, x)$ . Then, as a consequence of our results, we obtain the following:

*Let  $\mathcal{F} := \{F \in \text{Diff}_0(\mathbb{T}^2) \mid \pi_1 \circ F = \pi_1\}$ . Then there exists a non-empty set  $\mathcal{U} \subseteq \mathcal{F}$ , which is  $\mathcal{C}^1$ -open in  $\mathcal{F}$  and has the following property:*

*For any  $F \in \mathcal{U}$  there exists a set  $\Omega_F \subseteq \mathbb{T}^1$  of positive Lebesgue measure, such that for any  $\omega \in \Omega_F$  the map  $f = R_\omega \circ F$  is minimal and has a strange non-chaotic attractor.*

A more precise characterisation of the set  $\mathcal{U}$  in the above statement, in terms of explicit  $\mathcal{C}^1$ -estimates, is provided by Theorem 2.1 and/or Theorem 2.5 below.

Our second objective is to apply our methods to a particular model, which is well-known from the literature, namely the qpf Arnold circle map

$$(1.1) \quad (\theta, x) \mapsto (\theta + \omega, x + \tau + a \sin(2\pi x) + b \cos(2\pi\theta)^d) .$$

Here  $\tau \in \mathbb{T}^1$ ,  $a \in [0, 1/2\pi]$ ,  $b \in \mathbb{R}$  and  $d$  is an odd positive integer. This example was proposed by Ding *et al* [8] as a simple model of an oscillator forced at two or more incommensurate frequencies, and has been intensively studied numerically since<sup>1</sup> (see, for example, [9, 10, 11, 12, 13]). Provided  $d$  is chosen sufficiently large, we show that there exist rotation numbers  $\omega$  for which (1.1) exhibits SNA on a set of positive measure in the  $(\tau, a, b)$ -parameter space (see Corollary 2.8).

Particular attention in the study of (1.1) has been given to the structure of the Arnold tongues, which are subsets of the parameter space on which the rotation number stays constant. In [11], the authors observe that the Arnold tongue corresponding to rotation number zero seems to collapse in some regions of the parameter space. In Section 2.3, we prove that this happens at least for large  $d$ . In addition, we show that the boundaries of the zero tongue do not depend analytically on the parameter  $\beta$  in this case.

We want to mention that the approach employed here is inspired by the one of Bjerklöv in [6]. The latter was developed in the setting of quasiperiodic Schrödinger cocycles, but its techniques are basically non-linear, which allows us to adapt and to apply them to the non-linear setting. Similar ideas have also been used earlier by Young [7] to prove positive Lyapunov exponents for certain quasiperiodic  $\text{SL}(2, \mathbb{R})$ -cocycles.

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## 1.1 Notation

Let  $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$  and denote by  $\pi_i : \mathbb{T}^2 \rightarrow \mathbb{T}^1$  the projection to the respective coordinate. A *quasiperiodically forced (qpf) circle homeomorphism/diffeomorphism* is a homeomorphism/diffeomorphism  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  which is of the form

$$(1.2) \quad f : (\theta, x) \mapsto (\theta + \omega, f_\theta(x))$$

where  $\omega \in \mathbb{T}^1 \setminus \mathbb{Q}$  and the *fibre maps*  $f_\theta$  are defined by  $f_\theta(x) = \pi_2 \circ f(\theta, x)$ . Derivatives with respect to  $\theta$  or  $x$  will be denoted by  $\partial_\theta$  and  $\partial_x$ , respectively. Further, we use the notation

$$f_\theta^n(x) := \pi_2 \circ f^n(\theta, x) \quad \forall n \in \mathbb{Z} .$$

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<sup>1</sup>In the numerical studies usually  $d = 1$ . However, as mentioned in [8], any real-analytic forcing function is of more or less equal interest.

Note that this implies  $f_\theta^{-1} = (f_{\theta-\omega})^{-1}$ . For any  $a, b \in \mathbb{T}^1$ , we denote by

$$[a, b] := \{x \in \mathbb{T}^1 \mid a \leq x \leq b\}$$

the interval of all points  $x \in \mathbb{T}^1$  which lie between  $a$  and  $b$  in the counterclockwise direction, similarly for open intervals. Note that thus  $[b, a] = \mathbb{T}^1 \setminus (a, b)$ . For two points  $x, y \in \mathbb{T}^1$ , we denote the usual Euclidean distance on the circle by  $d(x, y)$ . We will also use the notation  $y - x$  in order to denote the distance between  $x$  and  $y$  in the counterclockwise direction, i.e. the length of the interval  $[x, y]$ .

If  $\varphi, \psi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  are two measurable functions, we let

$$[\varphi, \psi] := \{(\theta, x) \in \mathbb{T}^2 \mid x \in [\varphi(\theta), \psi(\theta)]\}$$

For any initial point  $(\theta_0, x_0) \in \mathbb{T}^2$  we denote its orbit by  $(\theta_k, x_k)_{k \in \mathbb{Z}}$ , that is

$$(\theta_k, x_k) := f^k(\theta_0, x_0) .$$

## 1.2 Some preliminaries

An *invariant graph* is a measurable function  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  which satisfies

$$f_\theta(\varphi(\theta)) = \varphi(\theta + \omega) \quad \forall \theta \in \mathbb{T}^1 .$$

This implies that the corresponding point set  $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in \mathbb{T}^1\}$  is  $f$ -invariant. The Lyapunov exponent of an invariant graph  $\varphi$  is defined as

$$\lambda(\varphi) = \int_{\mathbb{T}^1} \log |\partial_x f_\theta(\varphi(\theta))| d\theta .$$

We call a non-continuous invariant graph a *strange non-chaotic attractor* (SNA) if its Lyapunov exponent is negative and a *strange non-chaotic repeller* (SNR) if it is positive.

A convenient criterium for the existence of SNA involves pointwise Lyapunov exponents, forwards and backwards in time. These are given by

$$\lambda^+(\theta, x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\partial_x f_\theta^n(x)|$$

and

$$\lambda^-(\theta, x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\partial_x f_\theta^{-n}(x)| .$$

A point  $(\theta, x) \in \mathbb{T}^2$  (or more precisely its orbit) which has a positive Lyapunov exponent both forwards and backwards in time is called a *sink-source-orbit*. The existence of such orbits implies the existence of SNAs:

**Proposition 1.1** ([5]). *Suppose  $f$  is a quasiperiodically forced circle diffeomorphism which has a sink-source-orbit. Then  $f$  has both a SNA and a SNR.*

The proof in [5] is given for qpf monotone interval maps, but using [14, Theorem 4.1] it can easily be adapted to qpf circle diffeomorphisms.

The fibred rotation number of a qpf circle homeomorphism is defined as  $\rho(f) = \rho(F) \bmod 1$ , where  $F : \mathbb{T}^1 \times \mathbb{R} \hookrightarrow \mathbb{T}^1 \times \mathbb{R}$  is a lift of  $F$  and

$$(1.3) \quad \rho(F) := \lim_{n \rightarrow \infty} \frac{1}{n} (F_\theta^n(x) - x) .$$

This limit always exists and is independent of  $(\theta, x)$  [1]. Concerning the behaviour of the fibred rotation number with respect to strictly monotone perturbations, we will make use of the following:

**Proposition 1.2** ([15]). *Suppose a qpf circle homeomorphism  $f$  is minimal. Let  $F$  be a lift of  $f$  and  $F_\varepsilon(\theta, x) := (\theta + \omega, F_\theta(x) + \varepsilon)$ . Then the mapping  $\varepsilon \mapsto \rho(F_\varepsilon)$  is strictly monotone in  $\varepsilon = 0$ .*

In fact, the statement given in [15] is more general: The assertion of the proposition is true whenever  $f$  has no invariant strip, which is the appropriate analogue of a periodic orbit in this context (see [16] or [17] for the precise definition). Since invariant strips are always compact invariant strict subsets of  $\mathbb{T}^2$ , the above version follows immediately.

Finally, we will need a result concerning the uniqueness of the minimal set:

**Proposition 1.3** ([18]). *Suppose a qpf circle homeomorphism  $f$  is transitive. Then it has a unique minimal set.*

## 2 Main results

### 2.1 The existence of SNA and a first application

In the following, we will formulate a number of assumptions which are used in the statements of our main results. It is important to note that none of them involves the rotation number  $\omega$  on the base, since this will later be seen as a free parameter of the system. Thus, all the following conditions should be understood as assumptions on a collection of fibre maps  $(f_\theta)_{\theta \in \mathbb{T}^1}$ . Equivalently, the latter might be considered as a map  $F$  which satisfies  $\pi_1 \circ F = \text{Id}$ , as in highlighted statement in the introduction, such that  $F(\theta, x) = (\theta, f_\theta(x))$ .

*I. Regions in the phase space.* Suppose  $\mathcal{I}_0 \subseteq \mathbb{T}^1$  is a finite union of  $\mathcal{N}$  disjoint open intervals  $I_0^1, \dots, I_0^{\mathcal{N}}$ . We will refer to  $\mathcal{I}_0$  as the *first critical region*. Further, suppose that  $E = [e^-, e^+]$  and  $C = [c^-, c^+]$  are two non-empty, compact and disjoint intervals of positive length in  $\mathbb{T}^1$ . We will call  $E$  the *expanding* and  $C$  the *contracting interval*, motivated by the bounds on the derivatives given below. The first condition we require is a strong forward invariance of the contracting interval outside of the critical region:

$$(A1) \quad f_\theta(\text{cl}(\mathbb{T}^1 \setminus E)) \subseteq \text{int}(C) \quad \forall \theta \notin \mathcal{I}_0 .$$

Note that this implies

$$(A1') \quad f_\theta^{-1}(\text{cl}(\mathbb{T}^1 \setminus C)) \subseteq \text{int}(E) \quad \forall \theta \notin \mathcal{I}_0 + \omega .$$

*II. Bounds on the derivatives.* Let  $\alpha = (\alpha_l, \alpha_c, \alpha_e, \alpha_u) \in \mathbb{R}^4$  satisfy

$$0 < \alpha_l < \alpha_c < 1 < \alpha_e < \alpha_u$$

and suppose the following estimates hold:

$$(A2) \quad \alpha_l < \partial_x f_\theta(x) < \alpha_u \quad \forall (\theta, x) \in \mathbb{T}^2 ;$$

$$(A3) \quad \partial_x f_\theta(x) > \alpha_e \quad \forall (\theta, x) \in \mathbb{T}^1 \times E ;$$

$$(A4) \quad \partial_x f_\theta(x) < \alpha_c \quad \forall (\theta, x) \in \mathbb{T}^1 \times C .$$

$\alpha_e$  and  $\alpha_c$  will be referred to as the *expansion* and *contraction constants*,  $\alpha_l$  and  $\alpha_u$  as the *lower* and *upper bounds* (on the derivatives  $\partial_x f_\theta$ ).

Simply due to compactness, there also exists a global bound for the derivative w.r.t.  $\theta$ , i.e. a constant  $S > 0$  such that

$$(A5) \quad |\partial_\theta f_\theta(x)| < S \quad \forall (\theta, x) \in \mathbb{T}^2 .$$

*III. Transversal Intersections.* The last property we will need is the fact that for each connected component  $I_0^l$  of the first critical region  $\mathcal{I}_0$ , the set  $f(I_0^l \times C)$  crosses the expanding strip  $\mathbb{T}^1 \times E$  in a ‘nice’ transversal intersection, either upwards or downwards. This is ensured by the following: First, we suppose that

$$(A6) \quad |\partial_\theta f_\theta(x)| > s \quad \forall (\theta, x) \in \mathcal{I}_0 \times \mathbb{T}^1$$

for some constant  $s$  with  $0 < s < S$ . In particular, this implies that the sign of  $\partial_\theta f_\theta(x)$  is constant on every connected component  $I_0^l \times \mathbb{T}^1$  of  $\mathcal{I}_0 \times \mathbb{T}^1$ . We speak of an *upwards crossing* if it is positive and of a *downwards crossing* if it is negative. Secondly, we assume that

$$(A7) \quad \begin{aligned} &\exists! \theta_l^1 \in I_0^l \text{ with } f_{\theta_l^1}(c^+) = e^- \text{ and} \\ &\exists! \theta_l^2 \in I_0^l \text{ with } f_{\theta_l^2}(c^-) = e^+ . \end{aligned}$$

This ensures that the image of  $I_0^l \times C$  crosses the strip  $(I_0^l + \omega) \times E$  exactly once and does not ‘wind around the torus’ several times. Note that with respect to the canonical ordering inside the interval  $I_0^l$ , the point  $\theta_l^1$  lies on the right of  $\theta_l^2$  if the crossing is upwards and on the left of  $\theta_l^2$  if it is downwards.

Now we can state the first main result. The proof is given in Section 3.

**Theorem 2.1.** *Suppose  $(f_\theta)_{\theta \in \mathbb{T}^1}$  satisfies (A1)–(A7). Further assume that*

$$\alpha_c^{-1} = \alpha_e = \alpha^{\frac{2}{p}} \quad \text{and} \quad \alpha_l^{-1} = \alpha_u = \alpha^p$$

*for some  $p \in \mathbb{N}$ . Let  $\varepsilon_0 := \max_{l=1}^{\mathcal{N}} |I_0^l|$  and fix  $\delta > 0$ . Then there exists strictly positive constants  $c_0 = c_0(\delta, p, s, S, \mathcal{N})$  and  $\alpha_0 = \alpha_0(\delta, p, s, S, \mathcal{N})$  with the following property:*

*If  $\varepsilon_0 < c_0$  and  $\alpha > \alpha_0$ , then there exists a set  $\Omega \subseteq \mathbb{T}^1$  of measure*

$$\text{Leb}(\Omega) \geq 1 - \delta ,$$

*such that for all  $\omega \in \Omega$  the system*

$$(\theta, x) \mapsto (\theta + \omega, f_\theta(x))$$

*has a sink-source-orbit, and consequently a SNA and a SNR. In addition, the dynamics are minimal.*

**Remark 2.2.** (a) *Since all the conditions of the theorem are  $\mathcal{C}^1$ -open in  $\mathcal{F}$ , the highlighted statement in the introduction is an immediate consequence.*

(b) *Suppose that a qpf circle diffeomorphism  $f$  is minimal and has a SNA, as in the assertion of the theorem. Then it also has the property that its ‘deviations from the average rotation’*

$$(2.1) \quad F_\theta^n(x) - x - n\rho(F)$$

*are unbounded. This follows from a classification result for qpf circle homeomorphisms, which we want to discuss briefly.*

*If the quantities in (2.1) are uniformly bounded in  $n, \theta$  and  $x$ , then a direct analogue to Poincaré’s classification of circle homeomorphism holds [16]: Either  $f$  is semi-conjugate to an irrational torus translation, or there exists an invariant strip. The latter replace periodic orbits and are defined as compact invariant sets which intersect every fibre  $\{\theta\} \times \mathbb{T}^1$  in a finite number of intervals and have certain additional regularity properties (a precise definition is contained in [16] or [18]).*

Since  $f$  is minimal it cannot have an invariant strip (such sets are always strict subsets of the torus), since it has an SNA it cannot be semi-conjugate to an irrational torus translation (in this case there are no invariant graphs). Consequently, the two alternatives in the case of bounded deviations are ruled out, and the quantities in (2.1) have to be unbounded.

(c) There exists a mechanism for the creation of SNA which is very similar to the one studied here, but which leads to SNA which are the semi-continuous boundary graphs of invariant strips. In particular, the dynamics are not minimal and the deviations from the constant rotation (2.1) remain bounded. This mechanism is described in [19] and [5].

In order to give some explicit examples to which the above theorem applies, denote by  $\gamma : \mathbb{T}^1 \rightarrow (-1/2, 1/2)$  the lift of the identity map on  $\mathbb{T}^1$ . Then  $\pi \circ \gamma = \text{Id}_{\mathbb{T}^1}$ , where  $\pi : \mathbb{R} \rightarrow \mathbb{T}^1$  is the canonical projection. Further, given any  $p \geq 2$  define  $a : \mathbb{R} \rightarrow \mathbb{R}$  by

$$(2.2) \quad a_p(x) := \int_0^x \frac{1}{1 + |x|^p} dx .$$

Of course, for  $p = 2$  this just yields the arcus tangent. For a given parameter  $\alpha \in \mathbb{R}^+$  and  $x \in \mathbb{T}^1$ , let

$$(2.3) \quad h_\alpha(x) := \pi \left( \frac{a_p(\alpha \gamma(x))}{2a_p(\alpha/2)} \right) .$$

It is easy to check that for all  $\alpha$  the map  $h_\alpha$  is a diffeomorphism of the circle. Finally, let  $g \in \text{Diff}(\mathbb{T}^1)$  be such that

$$(2.4) \quad g^{-1}(\{1/2\}) \text{ is a finite and non-empty set ;}$$

$$(2.5) \quad g'(\theta) \neq 0 \quad \forall x \in g^{-1}(\{1/2\}) .$$

For example, one could choose  $g(\theta) = \beta \cos(2\pi\theta)$  for any  $\beta > \frac{1}{2}$ . Then Theorem 2.1 implies the following

**Corollary 2.3.** *Suppose  $h_\alpha$  and  $g$  are chosen as above and  $\delta > 0$  is fixed. Then there exists a constant  $\alpha_0 = \alpha_0(\delta, p, g)$  with the following property:*

*If  $\alpha \geq \alpha_0$ , then there exists a set  $\Omega \subseteq \mathbb{T}^1$  of measure  $\text{Leb}(\Omega) \geq 1 - \delta$ , such that for any  $\omega \in \Omega$  the system*

$$(2.6) \quad (\theta, x) \mapsto (\theta + \omega, h_\alpha(x) + g(\theta))$$

*has a sink-source-orbit and consequently a SNA and a SNR. In addition, the dynamics are minimal.*

The proof is given in Section 3.7 .

**Remark 2.4.** Let  $c_p := \lim_{x \rightarrow \infty} a_p(x)$  and suppose  $\tilde{h}_\alpha$  is the map which is obtained by projecting the mapping  $\mathbb{R} \leftrightarrow, x \mapsto \alpha^2 x$  to the circle via the change of variables  $x \mapsto \pi(a_p(x)/2c_p)$ . Then the preceding corollary remains true if  $h_\alpha$  is replaced by  $\tilde{h}_\alpha$ . The proof in Section 3.7 can be adapted easily.

However, in this case the map  $(\theta, x) \mapsto (\theta + \omega, \tilde{h}_\alpha(x) + g(\theta))$  is the projective action of the  $SL(2, \mathbb{R})$ -cocycle

$$\mathbb{T}^1 \times \mathbb{R}^2 \leftrightarrow, \quad (\theta, v) \mapsto (\theta + \omega, A(\theta)v)$$

with

$$A(\theta) = R_{g(\theta)} \circ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

where  $R_\phi$  denotes the rotation matrix with angle  $\phi$ . This means that, at least in the case of an analytic forcing function  $g$  and except for the minimality, similar statements can be derived from classical results on  $SL(2, \mathbb{R})$ -cocycles, for example in [1]. This is not true for the parameter family (2.6).

## 2.2 A refined result for the quasiperiodically forced Arnold circle map

The statement of Theorem 2.1 can be circumscribed by saying that SNA occur whenever the fibre maps are ‘sufficiently hyperbolic’, meaning that the expansion and contraction constants provided by (A3) and (A4) are large enough. However, concerning the forced Arnold circle map (1.1), this constitutes a problem. In the realm of invertibility, meaning for  $a \leq 1/2\pi$ , the derivative of the fibre maps is always bounded by 2. For the contraction, the situation is similar: While the derivative at  $x = \frac{1}{2}$  goes to zero as  $a$  goes to one, a strong contraction only takes place on a very small neighbourhood of the point  $\frac{1}{2}$ . For any interval of fixed length, the uniform contraction rate will always remain bounded.

In order to overcome this obstruction and to obtain a result which applies to the qpf Arnold circle map, we have to make use of additional information on the forcing function  $\theta \mapsto \cos(2\pi\theta)^d$ , namely of the fact that for large  $d$  its derivative almost vanishes on a large part of the phase space. This is done via the following assumption.

Suppose  $\mathcal{I}'_0 \subseteq \mathbb{T}^1$  is the disjoint union of at most  $\mathcal{N}$  open intervals and let  $s' \in (0, S)$ . Then assume that

$$(A8) \quad \mathcal{I}_0 \subseteq \mathcal{I}'_0 \quad \text{and} \quad |\partial_\theta f_\theta(x)| < s' \quad \forall (\theta, x) \in (\mathbb{T}^1 \setminus \mathcal{I}'_0) \times C.$$

The refined version of Theorem 2.1 now reads as follows:

**Theorem 2.5.** *Suppose  $(f_\theta)_{\theta \in \mathbb{T}^1}$  satisfies (A1)–(A8) and*

$$\alpha_c^{-1} = \alpha_e = \alpha^{\frac{2}{p}} \quad \text{and} \quad \alpha_l^{-1} = \alpha_u = \alpha^p$$

*for some  $p \in \mathbb{N}$ . Let  $\varepsilon_0 := \max_{l=1}^{\mathcal{N}} |I_0^l|$  and fix  $\delta > 0$ . Further, assume there exist constants  $A, d > 0$  such that*

$$(2.7) \quad S < A \cdot d,$$

$$(2.8) \quad s > \sqrt{d}/A,$$

$$(2.9) \quad \varepsilon_0 < A/\sqrt[3]{d}.$$

*Then there exist strictly positive constants  $c_0 = c_0(\delta, \alpha, p, \mathcal{N})$  and  $d_0 = d_0(\delta, \alpha, p, \mathcal{N}, A)$  with the following property:*

*If  $\frac{s'}{s} < c_0$  and  $d \geq d_0$ , then there exists a set  $\Omega \subseteq \mathbb{T}^1$  of measure*

$$\text{Leb}(\Omega) \geq 1 - \delta,$$

*such that for all  $\omega \in \Omega$  the system*

$$(\theta, x) \mapsto (\theta + \omega, f_\theta(x))$$

*has a sink-source-orbit and consequently a SNA and a SNR. In addition, the dynamics are minimal.*

Now suppose  $h$  is an orientation-preserving diffeomorphism of the circle, such that there exists disjoint closed intervals  $C, E \subseteq \mathbb{T}^1$  which satisfy

$$(2.10) \quad \sup_{x \in C} h'(x) < 1 \quad , \quad \inf_{x \in E} h'(x) > 1$$

and

$$(2.11) \quad h(\text{cl}(E^c)) \subseteq \text{int}(C).$$

For example, this holds whenever  $h$  has exactly two fixed points and exactly two points of inflexion.

**Corollary 2.6.** *Suppose  $h$  satisfies (2.10) and (2.11) and  $\delta > 0$  is fixed. Then there exist constants  $d_0 = d_0(\delta, h)$  and  $\varepsilon = \varepsilon(\delta, h)$  with the following property:*

*If  $d \geq d_0$  and  $b \in [1 - \varepsilon, 1 + \varepsilon]$ , then there exists a set  $\Omega \subseteq \mathbb{T}^1$  of measure  $\text{Leb}(\Omega) \geq 1 - \delta$ , such that for any  $\omega \in \Omega$  the system*

$$(2.12) \quad (\theta, x) \mapsto (\theta + \omega, h(x) + b \cos(2\pi\theta)^d)$$

*is minimal and has a SNA and a SNR.*

The proof is given in Section 4.2 .

**Remark 2.7.** (a) *Corollary 2.6 applies in particular to  $h(x) = x + \tau + a \sin(2\pi x)$  whenever  $0 \leq \tau < a < 1/2\pi$ . Thus, we obtain the existence of SNA for the qpf Arnold circle map (1.1). We reformulate the result in Corollary 2.8 below.*

(b) *We remark that the above statement remains true if  $\cos(2\pi\theta)^d$  is replaced by other forcing functions depending on a parameter  $d$ , as long as these show a similar scaling behaviour. For example, one could take  $g_d(\theta) = \left(\frac{1+\sin(2\pi\theta)}{2}\right)^d$ . In this case  $d \in \mathbb{R}^+$  can be chosen either very large or very small in order to ensure the existence of SNA. The proof of the corollary in Section 4.2 can be adapted accordingly. However, the symmetry  $\cos(2\pi(\theta + \frac{1}{2}))^d = -\cos(2\pi\theta)^d$  will play an important role in Section 2.3, such that we concentrate on this choice of the forcing function.*

In the literature, a typical point of view is to consider  $\omega$  and  $d$  as fixed and to view (1.1) as a three-parameter family depending on  $\tau, a$  and  $b$ . As a simple consequence of Fubini's Theorem we obtain

**Corollary 2.8.** *There is a constant  $d_0 > 0$ , such that for any  $d \geq d_0$  there exists a set of positive measure  $\Omega \subseteq \mathbb{T}^1$  with the following property:*

*For each  $\omega \in \Omega$  there exists a set of positive measure  $B_\omega \subseteq \mathbb{T}^1 \times [0, 1/2\pi] \times \mathbb{R}$ , such that for all  $(\tau, a, b) \in B_\omega$  the qpf Arnold circle map (1.1) is minimal and has a SNA and a SNR.*

Of course, similar statements hold if one likes to consider (1.1) as parameter family only depending on one or two parameters, while the other(s) are fixed.

## 2.3 Collapsing of the first Arnold tongue

In this section, we explain the consequences of our results for the structure of the first Arnold tongue. We denote the qpf Arnold circle map (1.1) with parameters  $\tau, a$  and  $b$  by  $f_{\tau,a,b}$ . First of all, the following statement is an immediate consequence of Corollary 2.6 applied to  $h(x) = x + a \sin(2\pi x)$  and Fubini's Theorem:

**Corollary 2.9.** *Given any  $a \in (0, 1/2\pi)$ , there exists a constant  $d_0 = d_0(a)$ , such that for any  $d \geq d_0$  there exists a set  $\Omega \subseteq \mathbb{T}^1$  of positive measure with the following property:*

*For any  $\omega \in \Omega$ , then there exists a set of positive measure  $B_\omega \subseteq \mathbb{R}$ , such that for any  $b \in B_\omega$  the qpf Arnold circle map  $f_{0,a,b}$  is minimal and has a SNA and a SNR.*

Since we want to study the dependence of the first Arnold tongue on the parameter  $b$ , the following notation will be convenient:

$$(2.13) \quad A_\rho^a := \{(\tau, b) \in \mathbb{T}^1 \times \mathbb{R} \mid \rho(f_{\tau,a,b}) = \rho\} .$$

As the rotation number depends monotonically on the parameter  $\tau$ , there exist functions  $\tau_{a,\rho}^-, \tau_{a,\rho}^+ : \mathbb{R} \rightarrow \mathbb{T}^1$ , such that

$$(2.14) \quad A_\rho^a = \{(\tau, b) \in \mathbb{T}^1 \times \mathbb{R} \mid \tau \in [\tau_{a,\rho}^-(b), \tau_{a,\rho}^+(b)]\} .$$



These functions  $\tau_{a,\rho}^\pm$  are continuous for all  $a, \rho$  and coincide (meaning  $\tau_{a,\rho}^- = \tau_{a,\rho}^+$ ) whenever  $\rho$  does not depend rationally on  $\omega$ , i.e.  $\rho \notin \mathbb{Q} + \mathbb{Q}\omega \bmod 1$  [15].

The canonical lift of the qpf Arnold circle map is given by

$$F_{\tau,a,b} : \mathbb{T}^1 \times \mathbb{R} \hookrightarrow \quad , \quad (\theta, x) \mapsto (\theta + \omega, x + \tau + a \sin(2\pi x) + b \cos(2\pi \theta)^d) .$$

Obviously there holds  $F_{0,a,b,\theta}(-x) = -F_{0,a,b,\theta}(x)$ . This symmetry immediately implies  $\rho(F_{0,a,b}) = 0$ , and therefore  $0 \in [\tau_{a,0}^-(b), \tau_{a,0}^+(b)] \forall b \in \mathbb{R}$ . On the other hand, if  $d \geq d_0(a)$  and  $b \in B_\omega$ , where  $d_0(a)$  and  $B_\omega$  are chosen as in the above corollary, then  $f_{0,a,b}$  is minimal. It therefore follows from Proposition 1.2 that the mapping  $\tau \mapsto \rho(f_{\tau,a,b})$  is strictly monotone at  $\tau = 0$ . Consequently, the first Arnold tongue is collapsed to a single point at this  $b$ -value, meaning  $\tau_{a,0}^-(b) = \tau_{a,0}^+(b) = 0$ . As this happens on a set of  $b$  of positive measure, and since the first Arnold tongue is clearly not collapsed at  $b = 0$ , the dependence of  $\tau_{a,0}^\pm$  on  $b$  cannot be real-analytic. We summarise our observations in the following

**Proposition 2.10.** *Suppose  $a \in (0, 1/2\pi)$  is fixed and  $d \geq d_0(a)$ , where  $d_0(a)$  is the constant provided by Corollary 2.9. Let  $B_\omega$  be as in the corollary.*

*Then for any  $b \in B_\omega$ , there holds  $\tau_{a,0}^-(b) = \tau_{a,0}^+(b) = 0$ . Furthermore, the mappings  $b \mapsto \tau_{a,0}^\pm(b)$  are not real-analytic.*

Of course, this raises the question whether the dependence of the boundaries of the Arnold tongues is analytic in  $a$ . We have to leave this open here. However, by the same arguments applied with the roles of  $a$  and  $b$  interchanged, one obtains the existence of parameters  $b$ , such that for a set of  $a$ 's of positive measure the first Arnold tongue is collapsed. Hence, if such a parameter  $b$  is fixed and the dependence on  $a$  was real-analytic, then the first tongue would have to be reduced to a single point for all  $a \in [0, 1/2\pi]$ .

### 3 Creation of SNA: The basic mechanism

The aim of this section is to prove Theorem 2.1. Thereby, we proceed in three steps. First, we place certain ‘imaginary’ conditions of the rotation number  $\omega$ , and show that these imply the existence of a sink-source-orbit (Sections 3.1 and 3.2). After this, it remains to show that there exist rotation numbers which satisfy these conditions. In order to do so, we first describe the geometry of certain critical sets, which were used before in the formulation of the conditions on  $\omega$  (Section 3.3). Using the obtained information, we then perform a parameter exclusion, which still leaves a set of positive measure of ‘good’  $\omega$ 's, which have all the required properties. The technical statements for the parameter exclusion are contained in Section 3.4, the final step in the proof is then given in Section 3.5. The proof of the minimality statement is contained in Section 3.6.

#### 3.1 Critical sets and good frequencies

*Critical sets.* First we have to define a sequence of *critical sets*, which project down to *critical regions* and play a major part in all that follows:

**Definition 3.1.** *For  $\omega \in \mathbb{T}^1$ ,  $\mathcal{I}_0$  as above and any monotonically increasing sequence  $(M_n)_{n \in \mathbb{N}_0}$  of integers with  $M_0 \geq 2$  we inductively define nested sequences  $\mathcal{C}_0, \mathcal{C}_1, \dots$  of critical sets and  $\mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \dots$  of critical regions in the following way: If  $\mathcal{I}_0, \dots, \mathcal{I}_n$  have been defined, let*

$$\begin{aligned} \mathcal{A}_n &:= \{(\theta, x) \mid \theta \in \mathcal{I}_n - (M_n - 1)\omega, x \in C\} , \\ \mathcal{B}_n &:= \{(\theta, x) \mid \theta \in \mathcal{I}_n + (M_n + 1)\omega, x \in E\} , \\ \mathcal{C}_n &:= f^{M_n-1}(\mathcal{A}_n) \cap f^{-M_n-1}(\mathcal{B}_n) \end{aligned}$$

and

$$\mathcal{I}_{n+1} := \text{int}(\pi_1(\mathcal{C}_n)) .$$

*Good frequencies.* Further, we impose certain ‘Diophantine’ conditions on the frequency  $\omega$ , which mainly state that the critical sets do not return too fast:

**Definition 3.2.** Suppose  $(M_n)_{n \in \mathbb{N}_0}$  and  $(\mathcal{I}_n)_{n \in \mathbb{N}_0}$  are chosen as above and let  $(K_n)_{n \in \mathbb{N}_0}$  be a monotonically increasing sequence of positive integers. Further, let  $(\varepsilon_n)_{n \in \mathbb{N}_0}$  be a non-increasing sequence of positive real numbers which satisfy  $\varepsilon_n \geq 3\varepsilon_{n+1} \ \forall n \in \mathbb{N}_0$ . Finally, let

$$\mathcal{X}_n := \bigcup_{k=1}^{2K_n M_n} (\mathcal{I}_n + k\omega) \quad \text{and} \quad \mathcal{Y}_n := \bigcup_{j=0}^n \bigcup_{k=-M_j+1}^{M_j+1} (\mathcal{I}_j + k\omega) .$$

Then we define  $\mathcal{F}_n = \mathcal{F}_n(M_0, \dots, M_n)$  as the set of those frequencies  $\omega \in \mathbb{T}^1$  which satisfy

$$(\mathcal{F}1)_n \quad d(\mathcal{I}_j, \mathcal{X}_j) > 3\varepsilon_j \quad \forall j = 0, \dots, n$$

and

$$(\mathcal{F}2)_n \quad d((\mathcal{I}_j - (M_j - 1)\omega) \cup (\mathcal{I}_j + (M_j + 1)\omega), \mathcal{Y}_{j-1}) > 0 \quad \forall j = 1, \dots, n .$$

Further, let

$$\mathcal{Z}_n := \bigcup_{j=0}^n \bigcup_{k=-M_j+2}^{M_j} (\mathcal{I}_j + k\omega) ,$$

$$\mathcal{Z}_{-1} := \emptyset \text{ and } \mathcal{F}_{-1} := \mathbb{T}^1 .$$

Finally let

$$\mathcal{V}_n := \bigcup_{j=0}^n \bigcup_{k=1}^{M_j+1} (\mathcal{I}_j + k\omega) \quad \text{and} \quad \mathcal{W}_n := \bigcup_{j=0}^n \bigcup_{k=-M_j+1}^0 (\mathcal{I}_j + k\omega)$$

$$\text{and } \mathcal{V}_{-1} = \mathcal{W}_{-1} = \emptyset .$$

**Remark 3.3.** For an easier reading of the following sections, the reader should keep in mind the following ‘intuitive’ description of the relations between the sets  $\mathcal{Y}_n$ ,  $\mathcal{Z}_n$ ,  $\mathcal{V}_n$  and  $\mathcal{W}_n$ :  $\mathcal{V}_n$  and  $\mathcal{W}_n$  are just the ‘right’ and ‘left’ part of  $\mathcal{Y}_n$ , whereas  $\mathcal{Z}_n$  is just reduced by one iterate on either side in comparison with  $\mathcal{Y}_n$ , such that  $\mathcal{Z}_n \pm \omega$  is still contained in  $\mathcal{Y}_n$ .

### 3.2 Construction of the sink-source-orbits

Recall that for any given point  $(\theta_0, x_0)$ , we denote its orbit by  $(\theta_k, x_k) = f^k(\theta_0, x_0)$ .

**Lemma 3.4.** Suppose (A1) holds. Then for all  $n \geq 0$ , the following are true:

Forwards iteration: If

$$(\mathcal{B}1)_n \quad \begin{cases} \omega & \in \mathcal{F}_{n-1} \\ \theta_0 & \notin \mathcal{Z}_{n-1} \\ x_0 & \in C \end{cases}$$

and  $\mathcal{L} \geq 0$  is the first integer, such that  $\theta_{\mathcal{L}} \in \mathcal{I}_n$ , then

$$(\mathcal{C}1)_n \quad x_m \notin C \Rightarrow \theta_m \in \mathcal{V}_{n-1} \quad \forall m = 1, \dots, \mathcal{L} .$$

Backwards iteration: If

$$(\mathcal{B}2)_n \quad \begin{cases} \omega & \in \mathcal{F}_{n-1} \\ \theta_0 & \notin \mathcal{Z}_{n-1} \\ x_0 & \in E \end{cases}$$

and  $\mathcal{R} \geq 0$  is the first integer, such that  $\theta_{-\mathcal{R}} \in \mathcal{I}_n + \omega$ , then

$$(\mathcal{C}2)_n \quad x_{-m} \notin E \Rightarrow \theta_{-m} \in \mathcal{W}_{n-1} \quad \forall m = 1, \dots, \mathcal{R}.$$

*Proof.* First of all, note that  $(\mathcal{C}1)_0$  follows directly from  $(\mathcal{A}1)$ . Now suppose that  $(\mathcal{B}1)_n$  implies  $(\mathcal{C}1)_n$  and fix  $\omega \in \mathcal{F}_n$ ,  $\theta_0 \notin \mathcal{Z}_n$  and  $x_0 \in C$ . Using  $(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$ , it is easy to see that

$$(3.1) \quad (\mathcal{I}_n - (M_n - 1)\omega) \cap \mathcal{V}_n = \emptyset,$$

$$(3.2) \quad (\mathcal{I}_n + (M_n + 1)\omega) \cap \mathcal{I}_0 = \emptyset,$$

$$(3.3) \quad (\mathcal{I}_n + (M_n + 2)\omega) \cap \mathcal{Z}_n = \emptyset.$$

Let  $\mathcal{L}$  be the first integer such that  $\theta_{\mathcal{L}} \in \mathcal{I}_{n+1}$  and let  $0 < L_1 < L_2 < \dots < L_J = \mathcal{L}$  be those times  $0 \leq i \leq \mathcal{L}$  with  $\theta_i \in \mathcal{I}_n$ . If we denote condition  $(\mathcal{C}1)_{n+1}$  with  $\mathcal{L}$  replaced by  $L_j$  by  $(\mathcal{C}1)_{n+1}[L_j]$ , then  $(\mathcal{C}1)_{n+1}[L_1]$  follows from  $(\mathcal{C}1)_n$  (note that  $\mathcal{Z}_{n-1} \subseteq \mathcal{Z}_n$ ,  $\mathcal{F}_n \subseteq \mathcal{F}_{n-1}$  and  $\mathcal{V}_{n-1} \subseteq \mathcal{V}_n$ ).

Assume now that  $(\mathcal{C}1)_{n+1}[L_j]$  holds for some  $1 \leq j < J$ . As  $\theta_0 \notin \mathcal{Z}_n$  we have  $L_j - M_n + 1 \geq 0$ , and as  $\theta_{L_j - M_n + 1} \notin \mathcal{V}_n$  due to (3.1) it follows that  $x_{L_j - M_n + 1} \in C$ . Consequently  $(\theta_{L_j - M_n + 1}, x_{L_j - M_n + 1}) \in \mathcal{A}_n$ , and as  $\theta_{L_j} \notin \mathcal{I}_{n+1}$  we must have  $(\theta_{L_j + M_n + 1}, x_{L_j + M_n + 1}) \notin \mathcal{B}_n$ , which means

$$x_{L_j + M_n + 1} \notin E.$$

As  $\theta_{L_j + M_n + 1} \notin \mathcal{I}_0$  by (3.2) we can apply  $(\mathcal{A}1)$  and obtain  $x_{L_j + M_n + 2} \in C$ . Before, we could have had  $x_k \notin C$  for some  $k \in \{L_j + 1, \dots, L_j + M_n + 1\}$ , but for such  $k$  there obviously holds

$$\theta_k \in \mathcal{I}_n + \omega \cup \dots \cup \mathcal{I}_n + (M_n + 1)\omega \subseteq \mathcal{V}_n.$$

Further, as  $\theta_{L_j + M_n + 2} \notin \mathcal{Z}_n \supseteq \mathcal{Z}_{n-1}$  by (3.3) and  $\mathcal{F}_n \subseteq \mathcal{F}_{n-1}$ , we can now apply  $(\mathcal{C}1)_n$  and obtain  $(\mathcal{C}1)_{n+1}[L_{j+1}]$ . As  $L_J = \mathcal{L}$ , this completes the proof of  $(\mathcal{C}1)_{n+1}$ .

Backwards iteration:  $(\mathcal{C}2)_0$  follows directly from  $(\mathcal{A}1')$ . Suppose that  $(\mathcal{B}1)_n$  implies  $(\mathcal{C}2)_n$  and fix  $\omega \in \mathcal{F}_n$ ,  $\theta_0 \notin \mathcal{Z}_n$  and  $x_0 \in E$ . Using  $(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$ , we see that

$$(3.4) \quad (\mathcal{I}_n + (M_n + 1)\omega) \cap \mathcal{W}_n = \emptyset,$$

$$(3.5) \quad (\mathcal{I}_n - (M_n - 1)\omega) \cap \mathcal{I}_0 = \emptyset,$$

$$(3.6) \quad (\mathcal{I}_n - M_n\omega) \cap \mathcal{Z}_n = \emptyset.$$

Let  $\mathcal{R}$  be the first integer such that  $\theta_{-\mathcal{R}} \in \mathcal{I}_{n+1} + \omega$  and let  $0 < R_1 < R_2 < \dots < R_J = \mathcal{R}$  be those times  $0 \leq i \leq \mathcal{R}$  with  $\theta_{-i} \in \mathcal{I}_n + \omega$ . If we denote condition  $(\mathcal{C}2)_{n+1}$  with  $\mathcal{R}$  replaced by  $R_j$  by  $(\mathcal{C}2)_{n+1}[R_j]$ , then  $(\mathcal{C}2)_{n+1}[R_1]$  follows from  $(\mathcal{C}2)_n$ .

Assume now that  $(\mathcal{C}2)_{n+1}[R_j]$  holds for some  $1 \leq j < J$ . As  $\theta_0 \notin \mathcal{Z}_n$  we have  $R_j - M_n \geq 0$ , and as  $\theta_{-R_j + M_n} \notin \mathcal{W}_n$  due to (3.4) (note that  $\theta_{-R_j - 1} \in \mathcal{I}_n$ ) it follows that  $x_{-R_j + M_n} \in E$ . Consequently  $(\theta_{-R_j + M_n}, x_{-R_j + M_n}) \in \mathcal{B}_n$ , and as  $\theta_{-R_j - 1} \notin \mathcal{I}_{n+1}$  we must have  $(\theta_{-R_j - M_n}, x_{-R_j - M_n}) \notin \mathcal{A}_n$ , which means

$$x_{-R_j - M_n} \notin C.$$

As  $\theta_{-R_j - M_n} \notin \mathcal{I}_0$  by (3.5) we can apply  $(\mathcal{A}1')$  and obtain  $x_{-R_j - M_n - 1} \in E$ . Before, we could have had  $x_{-k} \notin E$  for some  $k \in \{R_j + 1, \dots, R_j + M_n\}$ , but for such  $k$  there obviously holds

$$\theta_k \in \mathcal{I}_n \cup \mathcal{I}_n - \omega \cup \dots \cup \mathcal{I}_n - M_n\omega \subseteq \mathcal{W}_n.$$

Further, as  $\theta_{-R_j-M_{n-1}} \notin \mathcal{Z}_n \supseteq \mathcal{Z}_{n-1}$  by (3.6) and  $\mathcal{F}_n \subseteq \mathcal{F}_{n-1}$ , we can now apply  $(C2)_n$  and obtain  $(C2)_{n+1}[R_{j+1}]$ . As  $R_j = \mathcal{R}$ , this completes the proof.  $\square$

**Remark 3.5.** (a) Suppose  $(A1)$  holds,  $\omega \in \mathcal{F}_n$  and  $(\theta_0, x_0) \in \mathcal{A}_n$ . Then  $(B1)_n$  holds and  $\mathcal{L} = M_n - 1$ .

In order to see this, note that  $x_0 \in C$  holds by definition of  $\mathcal{A}_n$ , and  $\theta_0 \notin \mathcal{Z}_{n-1}$  follows from

$$(3.7) \quad (\mathcal{I}_n - (M_n - 1)\omega) \cap \mathcal{Z}_n = \emptyset ,$$

which is a consequence of see  $(F1)_n$  and  $(F2)_n$ .

(b) Similarly, suppose  $\omega \in \mathcal{F}_{n-1}$  and  $(\theta_0, x_0) \in \mathcal{B}_n$ . Then  $(B2)_n$  holds and  $\mathcal{R} = M_n$ .

This follows by the same argument as (a):  $x_0 \in E$  holds by definition of  $\mathcal{B}_n$  and  $\theta_0 \notin \mathcal{Z}_{n-1}$  follows from

$$(3.8) \quad (\mathcal{I}_n + (M_n + 1)\omega) \cap \mathcal{Z}_n = \emptyset ,$$

which is again a consequence of  $(F1)_n$  and  $(F2)_n$ .

**Corollary 3.6.** Suppose  $(A1)$  holds and  $\omega \in \mathcal{F}_n$ . Then

$$(3.9) \quad f^{M_n-M_{n-1}}(\mathcal{A}_n) \subseteq \mathcal{A}_{n-1} \quad \text{and} \quad f^{-M_n+M_{n-1}}(\mathcal{B}_n) \subseteq \mathcal{B}_{n-1} .$$

Consequently  $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots$ . Further

$$(3.10) \quad f^{M_n-1}(\mathcal{A}_n) \subseteq \mathcal{I}_n \times C \quad \text{and} \quad f^{-M_n}(\mathcal{B}_n) \subseteq (\mathcal{I}_n + \omega) \times E .$$

*Proof.* Let  $(\theta_0, x_0) \in \mathcal{A}_n$ , such that, by the preceding remark,  $(B1)_n$  holds. There holds

$$(\mathcal{I}_n - (M_{n-1} - 1)\omega) \cap \mathcal{V}_{n-1} = \emptyset .$$

This follows from (3.1), applied to  $n - 1$  and using that  $\mathcal{I}_n \subseteq \mathcal{I}_{n-1}$ . Therefore we have  $\theta_{M_n-M_{n-1}} \notin \mathcal{V}_{n-1}$ , such that we can apply Lemma 3.4 and obtain that  $x_{M_n-M_{n-1}} \in C$ , which means that  $f^{M_n-M_{n-1}}(\theta_0, x_0) \in \mathcal{A}_{n-1}$ . As  $(\theta_0, x_0) \in \mathcal{A}_n$  was arbitrary, this proves the first inclusion in (3.9), and the argument for the second one is similar. Finally, as

$$(3.11) \quad \mathcal{I}_n \cap \mathcal{V}_n = \emptyset$$

and

$$(3.12) \quad \mathcal{I}_n + \omega \cap \mathcal{W}_n = \emptyset$$

due to  $(F1)_n$ , the inclusions in (3.10) follow in the same way.  $\square$

The preceding lemma gives some first control about the time an orbit spends in the expanding and contracting region. In order to make use of this information, we need to quantify it. For given  $\omega, \theta_0, x_0$  and  $0 \leq m \leq N$  let

$$(3.13) \quad \mathcal{P}_m^N := \#\{k \in [m, N-1] \mid x_k \in C\} ,$$

$$(3.14) \quad \mathcal{Q}_m^N := \#\{k \in [m, N-1] \mid x_{-k} \in E\} .$$

Further, let  $\beta_0 = 1$  and

$$(3.15) \quad \beta_n := \prod_{j=0}^{n-1} \left(1 - \frac{1}{K_j}\right) .$$

**Lemma 3.7.** *Suppose (A1) holds. Then for all  $n \geq 0$  the following are true:*

Forwards iteration: *Suppose  $(B1)_n$  holds and let  $\mathcal{L}$  be chosen as in Lemma 3.4 . Then*

$$(C3)_n \quad \mathcal{P}_m^{\mathcal{L}} \geq \beta_n \cdot (\mathcal{L} - m) \quad \forall m = 0, \dots, \mathcal{L} - 1 .$$

*Further  $x_{\mathcal{L}} \in C$ .*

Backwards iteration: *Suppose  $(B2)_n$  holds and let  $\mathcal{R}$  be chosen as in Lemma 3.4 . Then*

$$(C4)_n \quad \mathcal{Q}_m^{\mathcal{R}} \geq \beta_n \cdot (\mathcal{R} - m) \quad \forall m = 0, \dots, \mathcal{R} - 1 .$$

*Further  $x_{-\mathcal{R}} \in E$ .*

*Proof.* As  $\mathcal{V}_{-1}$  is void,  $(C3)_0$  follows directly from  $(C1)_0$ . Suppose that  $(B1)_n$  implies  $(C3)_n$  and fix  $\omega \in \mathcal{F}_n$ ,  $\theta_0 \notin \mathcal{Z}_n$  and  $x_0 \in C$ . As in the proof of Lemma 3.4, let  $0 < L_1 < L_2 < \dots < L_J = \mathcal{L}$  be those times  $0 \leq i \leq \mathcal{L}$  with  $\theta_i \in \mathcal{I}_n$  and denote condition  $(C3)_n$  with  $\mathcal{L}$  replaced by  $L_j$  by  $(C3)_n[L_j]$ .

As  $\beta_{n+1} \leq \beta_n$ , condition  $(C3)_{n+1}[L_1]$  follows from  $(C3)_n$ . Suppose  $(C3)_{n+1}[L_j]$  holds for some  $1 \leq j < J$ . Using  $(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$  we see that

$$(3.16) \quad (\mathcal{I}_n + (M_n + 2)\omega) \cap \mathcal{V}_n = \emptyset ,$$

such that in particular  $\theta_{L_j+M_n+2} \notin \mathcal{V}_n$  and consequently  $x_{L_j+M_n+2} \in C$  by  $(C1)_{n+1}$ . As further  $\theta_{L_j+M_n+2} \notin \mathcal{Z}_n$  by (3.3),  $(C3)_n$  implies that for any  $m \in [L_j + M_n + 2, L_{j+1}]$  there holds

$$(3.17) \quad \mathcal{P}_m^{L_{j+1}} \geq \beta_n \cdot (L_{j+1} - m) .$$

This proves  $(C3)_{n+1}[L_{j+1}]$  for such  $m$ . Further, by  $(\mathcal{F}1)_n$  we have  $L_{j+1} - L_j > 2K_n M_n$ . Hence, for any  $m \in [L_j, L_j + M_n + 1]$  we obtain the estimate

$$\begin{aligned} \mathcal{P}_m^{L_{j+1}} &\geq \mathcal{P}_{L_j+M_n+2}^{L_{j+1}} \geq \beta_n \cdot (L_{j+1} - L_j - M_n - 2) \\ &\geq \beta_n \cdot \frac{L_{j+1} - L_j - M_n - 2}{L_{j+1} - L_j} \cdot (L_{j+1} - m) \\ &\geq \beta_n \cdot \left(1 - \frac{M_n + 2}{2K_n M_n}\right) (L_{j+1} - m) \stackrel{M_0 \geq 2}{\geq} \beta_{n+1} \cdot (L_{j+1} - m) . \end{aligned}$$

Finally, if  $m \in [0, L_j]$  the statement follows by combining the estimate for  $\mathcal{P}_{L_j}^{L_{j+1}}$  with the one for  $\mathcal{P}_m^{L_j}$  obtained from  $(C3)_{n+1}[L_j]$ .

Backwards iteration: As  $\mathcal{W}_{-1}$  is void,  $(C4)_0$  follows directly from  $(C2)_0$ . Suppose that  $(B2)_n$  implies  $(C4)_n$  and fix  $\omega \in \mathcal{F}_n$ ,  $\theta_0 \notin \mathcal{Z}_n$  and  $x_0 \in E$ . Let  $0 < R_1 < R_2 < \dots < R_J = \mathcal{R}$  be those times  $0 \leq i \leq \mathcal{L}$  with  $\theta_{-i} \in \mathcal{I}_n + \omega$  and denote condition  $(C4)_n$  with  $\mathcal{R}$  replaced by  $R_j$  by  $(C4)_n[R_j]$ .

As  $\beta_{n+1} \leq \beta_n$ , condition  $(C4)_{n+1}[R_1]$  follows from  $(C4)_n$ . Suppose  $(C4)_{n+1}[R_j]$  holds for some  $1 \leq j < J$ . Using  $(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$  we see that

$$(3.18) \quad (\mathcal{I}_n - M_n \omega) \cap \mathcal{W}_n = \emptyset ,$$

such that in particular  $\theta_{-R_j-M_n-1} \notin \mathcal{W}_n$  and consequently  $x_{-R_j-M_n-1} \in E$  by  $(C2)_{n+1}$ . As further  $\theta_{-R_j-M_n-1} \notin \mathcal{Z}_n$  by (3.6),  $(C4)_n$  implies that for any  $m \in [R_j + M_n + 1, R_{j+1}]$  there holds

$$(3.19) \quad \mathcal{Q}_m^{R_{j+1}} \geq \beta_n \cdot (R_{j+1} - m) .$$

This proves  $(\mathcal{C}4)_{n+1}[R_{j+1}]$  for such  $m$ . Further, by  $(\mathcal{F}1)_n$  we have  $R_{j+1} - R_j > 2K_n M_n$ . Hence, for any  $m \in [R_j, R_j + L_n]$  we obtain the estimate

$$\begin{aligned} \mathcal{Q}_m^{R_{j+1}} &\geq \mathcal{Q}_{R_j+M_n+1}^{R_{j+1}} \geq \beta_n \cdot (R_{j+1} - R_j - M_n - 1) \\ &\geq \beta_n \cdot \frac{R_{j+1} - R_j - M_n - 1}{R_{j+1} - R_j} \cdot (R_{j+1} - m) \\ &\geq \beta_n \cdot \left(1 - \frac{M_n + 1}{2K_n M_n}\right) (R_{j+1} - m) \geq \beta_{n+1} \cdot (R_{j+1} - m). \end{aligned}$$

Finally, if  $m \in [0, R_j]$  the statement follows by combining the estimate for  $\mathcal{P}_{R_j}^{R_{j+1}}$  with the one that  $(\mathcal{C}4)_{n+1}[R_j]$  yields for  $\mathcal{P}_m^{R_j}$ .  $\square$

Let

$$(3.20) \quad \beta := \lim_{n \rightarrow \infty} \beta_n = \inf_n \beta_n$$

and

$$(3.21) \quad \alpha_- := \alpha_e^\beta \alpha_u^{1-\beta}, \quad \alpha_+ := \alpha_e^\beta \alpha_l^{1-\beta}.$$

**Corollary 3.8.** *Suppose (A1)–(A4) hold and  $\omega \in \mathcal{F}_n$ . If  $(\theta, x) \in \text{cl}(f^{M_n}(\mathcal{A}_n))$ , then for all  $k \in [0, M_n]$  there holds*

$$(3.22) \quad \partial_x f_\theta^{-k}(x) \geq \alpha_-^{-k}.$$

*If  $(\theta, x) \in \text{cl}(f^{-M_n}(\mathcal{B}_n))$ , then for all  $k \in [0, M_n]$  there holds*

$$(3.23) \quad \partial_x f_\theta^k(x) \geq \alpha_+^k.$$

*Proof.* By continuity, it suffices to prove the above estimates on  $f^{M_n}(\mathcal{A}_n)$  and  $f^{-M_n}(\mathcal{B}_n)$ , respectively. We start by proving (3.23).

Suppose  $(\theta, x) \in f^{-M_n}(\mathcal{B}_n)$  and let  $(\theta_0, x_0) = f^{M_n}(\theta, x) \in \mathcal{B}_n$ . Then due to Remark 3.5 we have that  $\mathcal{R} = M_n$  and  $(\mathcal{B}2)_n$  holds. Using (A2), (A3) and the fact that  $x = x_{-\mathcal{R}} \in E$  (see Lemma 3.7) we obtain

$$(3.24) \quad \partial_x f_\theta^k(x) = \prod_{j=\mathcal{R}-k+1}^{\mathcal{R}} \partial_x f_{\theta_{-j}}(x_{-j}) \geq \alpha_e \cdot \alpha_e^{\mathcal{Q}_{\mathcal{R}-k+1}^{\mathcal{R}}} \cdot \alpha_l^{k-1-\mathcal{Q}_{\mathcal{R}-k+1}^{\mathcal{R}}}$$

Applying  $(\mathcal{C}4)_n$  and using that  $\alpha_e \geq \alpha_+$  yields the statement.

As  $\partial_x f_\theta^{-k}(x) = (\partial_x f_{\theta-k\omega}(f_\theta^{-k}(x)))^{-1}$ , the estimate in (3.22) can be obtained in the same way.  $\square$

**Proposition 3.9.** *Suppose (A1)–(A4) hold,  $\min\{\alpha_-^{-1}, \alpha_+\} > 1$ ,  $\omega \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_n$  and all critical sets  $\mathcal{I}_n$  are non-void. Then  $f$  has a sink-source-orbit.*

*Proof.* As all critical sets  $\mathcal{I}_n$  are non-void, the same is obviously true for the sets  $\text{cl}(\mathcal{C}_n) =$  and their images  $\text{cl}(f(\mathcal{C}_n)) = \text{cl}(f^{M_n}(\mathcal{A}_n)) \cap \text{cl}(f^{-M_n}(\mathcal{B}_n))$ . Due to Corollary 3.6, the later form a nested sequence of compact sets, such that their intersection is non-void as well. Let  $(\theta, x) \in \bigcap_{n \in \mathbb{N}} \text{cl}(f(\mathcal{C}_n))$ . Then due to (3.22) and as  $M_n \nearrow \infty$ , we obtain

$$\lambda^-(\theta, x) = \limsup_{k \rightarrow \infty} \frac{1}{k} \log |\partial_x f_\theta^{-k}(x)| \geq -\log \alpha_- > 0.$$

and similarly (3.23) yields  $\lambda^+(\theta, x) \geq \log \alpha_+ > 0$ .  $\square$

### 3.3 Geometry of the critical sets

In this section we turn to the description of the critical sets  $\mathcal{C}_n$  and the corresponding critical regions  $\mathcal{I}_{n+1}$ . In particular, we want to obtain information about their size and their dependence on  $\omega$  (which we have kept implicit so far). Suppose  $I = I(\omega) = (a(\omega), b(\omega))$  is a connected component of  $\mathcal{I}_n$ . Then we use the notation

$$|\partial_\omega I| = \max\{|\partial_\omega a(\omega)|, |\partial_\omega b(\omega)|\},$$

provided both derivatives on the right side exist. In this case we call  $I$  differentiable with respect to  $\omega$ . We will use the following inductive assumption:

$$(\mathcal{I})_n \quad \left\{ \begin{array}{ll} (i) & \text{For each } j \in [0, n], \mathcal{I}_j \text{ consists of } \mathcal{N} \text{ disjoint open intervals} \\ & I_j^1, \dots, I_j^{\mathcal{N}}. \\ (ii) & \text{For } j \in [1, n], \text{ each connected component of } \mathcal{I}_{j-1} \text{ contains} \\ & \text{exactly one connected component of } \mathcal{I}_j. \text{ Thus, by suitable} \\ & \text{labelling, } I_j^\iota \subseteq I_{j-1}^\iota \ \forall \iota = 1, \dots, \mathcal{N}. \\ (iii) & \text{For all } j \in [0, n] \text{ the set } \mathcal{F}_j \text{ is open and all } I_j^\iota \text{ are differentiable} \\ & \text{with respect to } \omega \text{ on } \mathcal{F}_j. \end{array} \right.$$

Note  $(\mathcal{I})_0$  follows directly from the choice of  $\mathcal{I}_0$  in Section 2.1 and the definition of  $\mathcal{F}_0$ . (The second statement is void for  $n = 0$ .)

In order to describe the geometry of the critical sets  $\mathcal{C}_n$ , or rather their images  $f(\mathcal{C}_n)$ , we have to introduce some notation and make some preliminary remarks, which we will use in the whole section. For any  $\iota \in [1, \mathcal{N}]$  we let

$$(3.25) \quad \mathcal{A}_n^\iota := \{(\theta, x) \mid \theta \in I_n^\iota - (M_n - 1)\omega, x \in C\},$$

$$(3.26) \quad \mathcal{B}_n^\iota := \{(\theta, x) \mid \theta \in I_n^\iota + (M_n + 1)\omega, x \in E\}.$$

For  $\theta \in \mathcal{I}_n + \omega$  let

$$(3.27) \quad \varphi_{\iota,n}^\pm(\theta) := f_{\theta-M_n\omega}^{M_n}(c^\pm) \quad \text{and} \quad \psi_{\iota,n}^\pm(\theta) := f_{\theta+M_n\omega}^{-M_n}(e^\pm),$$

such that

$$\begin{aligned} f^{M_n}(\mathcal{A}_n^\iota) &= \{(\theta, x) \mid \theta \in I_n^\iota + \omega, x \in [\varphi_{\iota,n}^-(\theta), \varphi_{\iota,n}^+(\theta)]\}, \\ f^{-M_n}(\mathcal{B}_n^\iota) &= \{(\theta, x) \mid \theta \in I_n^\iota + \omega, x \in [\psi_{\iota,n}^-(\theta), \psi_{\iota,n}^+(\theta)]\}. \end{aligned}$$

In order to start the induction, it is also convenient to define

$$(3.28) \quad \varphi_{-1}^\pm(\theta) := f_{\theta-\omega}(c^\pm) \quad \text{and} \quad \psi_{-1}^\pm(\theta) := e^\pm.$$

In all of the proofs of this section we will always fix  $\iota$  in order to concentrate on one connected component of  $\mathcal{I}_n$ . In principle we would have to distinguish two cases, namely that of an upwards and that of a downwards crossing (see (A7)). However, as the two cases are completely symmetric we can always assume, without loss of generality, that the crossing between  $f^{M_n}(\mathcal{A}_n)$  and  $f^{-M_n}(\mathcal{B}_n)$  is ‘upwards’, that is  $\partial_\theta f_\theta(x) > s$  on  $I_n^\iota \subseteq I_0^\iota$ .

Then the second inductive assumption which will be used in this section is the following: Suppose that  $I_n^\iota(\omega) = (a_{\iota,n}(\omega), b_{\iota,n}(\omega))$  and let  $J_n^\varphi(\theta) := (\varphi_n^-(\theta), \varphi_n^+(\theta))$  and  $J_n^\psi(\theta) := (\psi_n^-(\theta), \psi_n^+(\theta))$ . Then we will assume that

$$(\Phi/\Psi)_n \quad \begin{aligned} J_{n-1}^\varphi(a_{\iota,n}(\omega) + \omega) \cap J_{n-1}^\psi(a_{\iota,n}(\omega) + \omega) &= \emptyset \\ J_{n-1}^\varphi(b_{\iota,n}(\omega) + \omega) \cap J_{n-1}^\psi(b_{\iota,n}(\omega) + \omega) &= \emptyset \end{aligned}$$

Note that due to the definition of  $\varphi_{-1}^{\pm}$  and  $\psi_{-1}^{\pm}$  in (3.28), the statement  $(\Phi/\Psi)_0$  is a consequence of (A1).

Now we can derive some estimates concerning the geometry of the sets  $f(\mathcal{C}_n)$ . We start with an easy one. Let

$$(3.29) \quad h_n^{\varphi} := \inf_{\theta \in \mathcal{I}_n + \omega} |\varphi_n^+(\theta) - \varphi_n^-(\theta)| ,$$

$$(3.30) \quad h_n^{\psi} := \inf_{\theta \in \mathcal{I}_n + \omega} |\psi_n^+(\theta) - \psi_n^-(\theta)| ,$$

$$(3.31) \quad H_n^{\varphi} := \sup_{\theta \in \mathcal{I}_n + \omega} |\varphi_n^+(\theta) - \varphi_n^-(\theta)| ,$$

$$(3.32) \quad H_n^{\psi} := \sup_{\theta \in \mathcal{I}_n + \omega} |\psi_n^+(\theta) - \psi_n^-(\theta)| .$$

**Lemma 3.10.** *Suppose (A1)–(A4) hold and  $\omega \in \mathcal{F}_n$ . Then*

$$(3.33) \quad |C| \cdot \alpha_l^{M_n} \leq h_n^{\varphi} \leq H_n^{\varphi} \leq |C| \cdot \alpha_-^{M_n}$$

and

$$(3.34) \quad |E| \cdot \alpha_u^{-M_n} \leq h_n^{\psi} \leq H_n^{\psi} \leq |E| \cdot \alpha_+^{-M_n} .$$

*Proof.* As the vertical size of the sets  $\mathcal{A}_n$  and  $\mathcal{B}_n$  is  $|C|$  and  $|E|$ , respectively, the lower bounds are a direct consequence of (A2) and the upper bounds follow from Corollary 3.8.  $\square$

Next, we turn to some more serious estimates. Let

$$(3.35) \quad l_n^{\varphi} := \inf_{\theta \in \mathcal{I}_n + \omega} |\partial_{\theta} \varphi_n^{\pm}(\theta)| ,$$

$$(3.36) \quad u_n^{\varphi} := \sup_{\theta \in \mathcal{I}_n + \omega} |\partial_{\theta} \varphi_n^{\pm}(\theta)| ,$$

$$(3.37) \quad u_n^{\psi} := \sup_{\theta \in \mathcal{I}_n + \omega} |\partial_{\theta} \psi_n^{\pm}(\theta)| .$$

**Lemma 3.11.** *Suppose (A1)–(A7) hold and  $\omega \in \mathcal{F}_n$ . Then*

$$(3.38) \quad s - S/(\alpha_-^{-1} - 1) \leq l_n^{\varphi} \leq u_n^{\varphi} \leq S + S/(\alpha_-^{-1} - 1)$$

and

$$(3.39) \quad u_n^{\psi} \leq S/(\alpha_+ - 1) .$$

*Proof.* In order to prove (3.38), note that for any  $\mathcal{L} \in \mathbb{N}$  and  $(\theta_0, x_0) \in \mathbb{T}^2$  there holds

$$(3.40) \quad \partial_{\theta} f_{\theta_0}^{\mathcal{L}+1}(x_0) = \partial_{\theta} f_{\theta_1}^{\mathcal{L}}(x_1) + \partial_x f_{\theta_1}^{\mathcal{L}}(x_1) \cdot \partial_{\theta} f_{\theta_0}(x_0) .$$

By induction, we thus obtain

$$(3.41) \quad \partial_{\theta} f_{\theta_0}^{\mathcal{L}+1}(x_0) = \partial_{\theta} f_{\theta_{\mathcal{L}}}(x_{\mathcal{L}}) + \sum_{k=0}^{\mathcal{L}-1} \partial_x f_{\theta_{k+1}}^{\mathcal{L}-k}(x_{k+1}) \cdot \partial_{\theta} f_{\theta_k}(x_k) .$$

Now suppose  $\theta \in I_n^t + \omega$  and let  $(\theta_0, x_0) = (\theta - M_n \omega, c^{\pm})$  and  $\mathcal{L} = M_n - 1$ , such that  $f_{\theta_0}^{\mathcal{L}+1}(x_0) = \varphi_n^{\pm}(\theta)$ . Note that thus  $\mathcal{L}$  coincides with the choice in Lemma 3.4 (see Remark 3.5). By (A5) and (A6) we have

$$s < |\partial_{\theta} f_{\theta_{\mathcal{L}}}(x_{\mathcal{L}})| < S .$$



Further, using (3.22) from Corollary 3.8 we obtain that

$$(3.42) \quad \left| \partial_x f_{\theta_{k+1}}^{\mathcal{L}-k}(x_{k+1}) \right| = \left| \left( \partial_x f_{\theta_{\mathcal{L}+1}}^{-(\mathcal{L}-k)}(x_{\mathcal{L}+1}) \right)^{-1} \right| \leq \alpha_-^{\mathcal{L}-k}.$$

As  $|\partial_\theta f_{\theta_k}| \leq S \forall k$  by (A5), this yields the required estimates.

The proof of (3.39) is slightly more intricate. First of all, similar to (3.41) we obtain that for any  $\mathcal{R} \in \mathbb{N}$  and  $(\theta_0, x_0) \in \mathbb{T}^2$

$$(3.43) \quad \partial_\theta f_{\theta_0}^{-\mathcal{R}}(x_0) = \sum_{k=1}^{\mathcal{R}} \partial_x f_{\theta_{-k}}^{-\mathcal{R}+k}(x_{-k}) \cdot \partial_\theta f_{\theta_{-k+1}}^{-1}(x_{-k+1}).$$

Let  $(\theta_0, x_0) = (\theta + M_n \omega, e^\pm)$  and  $\mathcal{R} = M_n$ , such that  $f_{\theta_0}^{-\mathcal{R}}(x_0) = \psi_n^\pm(\theta)$ . Again, this coincides with the choice of  $\mathcal{R}$  in Lemma 3.4. In order to obtain an estimate on the second factor in the sum in (3.43), we note that

$$0 = \partial_\theta (f_{\theta-\omega} \circ f_\theta^{-1}(x)) = \partial_\theta f_{\theta-\omega}(f_\theta^{-1}(x)) + \partial_x f_{\theta-\omega}(f_\theta^{-1}(x)) \cdot \partial_\theta f_\theta^{-1}(x),$$

such that

$$(3.44) \quad |\partial_\theta f_\theta^{-1}(x)| \leq \left| \frac{\partial_\theta f_{\theta-\omega}(f_\theta^{-1}(x))}{\partial_x f_{\theta-\omega}(f_\theta^{-1}(x))} \right|.$$

Therefore  $|\partial_\theta f_{\theta_{-k+1}}^{-1}(x_{-k+1})|$  will be smaller than  $\frac{|\partial_\theta f_{\theta_{-k}}(x_{-k})|}{\alpha_e}$  whenever  $x_{-k} \in E$  and always smaller than  $\frac{|\partial_\theta f_{\theta_{-k}}(x_{-k})|}{\alpha_l}$ . Combining this with (3.24) yields

$$(3.45) \quad \begin{aligned} & \left| \partial_x f_{\theta_{-k}}^{-\mathcal{R}+k}(x_{-k}) \cdot \partial_\theta f_{\theta_{-k+1}}^{-1}(x_{-k+1}) \right| = \\ & = \left| \left( \partial_x f_{\theta_{-\mathcal{R}}}^{\mathcal{R}-k}(x_{-\mathcal{R}}) \right)^{-1} \cdot \partial_\theta f_{\theta_{-k+1}}^{-1}(x_{-k+1}) \right| \\ & \leq \alpha_e^{-1} \cdot \alpha_e^{-\mathcal{Q}_k^{\mathcal{R}}} \cdot \alpha_l^{-(\mathcal{R}-k-\mathcal{Q}_k^{\mathcal{R}})} \cdot |\partial_\theta f_{\theta_{-k}}(x_{-k})| \\ & \leq \alpha_+^{-(\mathcal{R}+1-k)} \cdot |\partial_\theta f_{\theta_{-k}}(x_{-k})| \leq \alpha_+^{-(\mathcal{R}+1-k)} \cdot S, \end{aligned}$$

and summing up over  $k$  proves (3.39).  $\square$

For the remainder of this section, we will write  $\varphi_n^\pm(\theta) = \varphi_n^\pm(\theta, \omega)$  and  $\psi_n^\pm(\theta) = \psi_n^\pm(\theta, \omega)$ , in order to make the dependence on  $\omega$  explicit. Let

$$(3.46) \quad \gamma_n^\varphi := \sup_{\theta \in \mathcal{I}_n + \omega} \left| \partial_\theta \varphi_n^\pm(\theta, \omega) + \partial_\omega \varphi_n^\pm(\theta, \omega) \right|,$$

$$(3.47) \quad \gamma_n^\psi := \sup_{\theta \in \mathcal{I}_n + \omega} \left| \partial_\theta \psi_n^\pm(\theta, \omega) + \partial_\omega \psi_n^\pm(\theta, \omega) \right|.$$

**Lemma 3.12.** *Suppose (A1)–(A7) hold and  $\omega \in \mathcal{F}_n$ . Then*

$$(3.48) \quad \gamma_n^\varphi \leq S \cdot \sum_{k=1}^{\infty} k \alpha_-^k$$

and

$$(3.49) \quad \gamma_n^\psi \leq S \cdot \sum_{k=1}^{\infty} (k+1) \alpha_+^{-k}.$$

*Proof.* For any  $k, \mathcal{L} \in \mathbb{N}$   $(\theta, x) \in \mathbb{T}^2$  there holds

$$(3.50) \quad \begin{aligned} \partial_\omega f_{\theta-(\mathcal{L}+1)\omega}^{k+1}(x) &= -(\mathcal{L}+1-k) \cdot \partial_\theta f_{\theta-(\mathcal{L}+1-k)\omega}(f_{\theta-(\mathcal{L}+1)\omega}^k(x)) \\ &+ \partial_x f_{\theta-(\mathcal{L}+1-k)\omega}(f_{\theta-(\mathcal{L}+1)\omega}^k(x)) \cdot \partial_\omega f_{\theta-(\mathcal{L}+1)\omega}^k(x) . \end{aligned}$$

As in the preceding proof, let  $(\theta_0, x_0) = (\theta - M_n \omega, c^\pm)$  and  $\mathcal{L} = M_n - 1$ . Then (3.50) simplifies to

$$(3.51) \quad \partial_\omega f_{\theta_0}^{k+1}(x_0) = -(\mathcal{L}+1-k) \cdot \partial_\theta f_{\theta_k}(x_k) + \partial_x f_{\theta_k}(x_k) \cdot \partial_\omega f_{\theta_0}^k(x_0) ,$$

and inductive application gives

$$(3.52) \quad \partial_\omega f_{\theta_0}^{\mathcal{L}+1}(x_0) = -\partial_\theta f_{\theta_\mathcal{L}}(x_\mathcal{L}) - \sum_{k=0}^{\mathcal{L}-1} (\mathcal{L}+1-k) \cdot \partial_x f_{\theta_{k+1}}^{\mathcal{L}-k}(x_{k+1}) \cdot \partial_\theta f_{\theta_k}(x_k) .$$

Combining this with (3.41) and using (3.42) yields

$$(3.53) \quad |\partial_\theta \varphi_n^\pm(\theta, \omega) + \partial_\omega \varphi_n^\pm(\theta, \omega)| = |\partial_\theta f_{\theta_0}^{\mathcal{L}+1}(x_0) + \partial_\omega f_{\theta_0}^{\mathcal{L}+1}(x_0)|$$

$$(3.54) \quad = \left| \sum_{k=0}^{\mathcal{L}-1} (\mathcal{L}-k) \cdot \partial_x f_{\theta_{k+1}}^{\mathcal{L}-k}(x_{k+1}) \cdot \partial_\theta f_{\theta_k}(x_k) \right|$$

$$(3.55) \quad \leq \sum_{k=0}^{\mathcal{L}-1} (\mathcal{L}-k) \cdot \alpha_-^{\mathcal{L}-k} \cdot S \leq S \cdot \sum_{k=1}^{\infty} k \alpha_-^k .$$

This proves (3.48).

Now let  $(\theta_0, x_0) = (\theta + M_n \omega, e^\pm)$  and  $\mathcal{R} = M_n$ . Similar to (3.52) there holds

$$(3.56) \quad \partial_\omega f_{\theta_0}^{-\mathcal{R}}(x_0) = \sum_{k=0}^{\mathcal{R}-1} (\mathcal{R}-k) \cdot \partial_x f_{\theta_{-k-1}}^{-\mathcal{R}+k+1}(x_{-k-1}) \cdot \partial_\theta f_{\theta_{-k}}^{-1}(x_{-k}) .$$

Using (3.45) as in the proof of Lemma 3.11 we obtain

$$\begin{aligned} |\partial_\omega \psi_n^\pm(\theta, \omega)| &= |\partial_\omega f_{\theta_0}^{-\mathcal{R}}(x_0)| \\ &\leq \sum_{k=0}^{\mathcal{R}-1} (\mathcal{R}-k) \cdot \alpha_+^{-(\mathcal{R}-k)} \cdot S \leq S \cdot \sum_{k=1}^{\infty} k \alpha_+^{-k} . \end{aligned}$$

Combined with (3.39), this yields (3.49).  $\square$

**Lemma 3.13.** *Suppose that (A1) holds and  $\omega \in \mathcal{F}_n$ . Further assume that  $(\mathcal{I})_n$  and  $(\Phi/\Psi)_n$  hold and  $l_n^\varphi > u_n^\psi$ . Then  $(\mathcal{I})_{n+1}$  and  $(\Phi/\Psi)_{n+1}$  hold and for all  $\iota = 1, \dots, \mathcal{N}$ . In addition*

$$(3.57) \quad \frac{h_n^\varphi + h_n^\psi}{u_n^\varphi + u_n^\psi} \leq |I_{n+1}^\iota| \leq \frac{H_n^\varphi + H_n^\psi}{l_n^\varphi - u_n^\psi}$$

and

$$(3.58) \quad |\partial_\omega I_{n+1}^\iota| \leq \frac{\gamma_n^\varphi + \gamma_n^\psi}{l_n^\varphi - u_n^\psi} .$$

*Proof.* As  $f^{M_n}(\mathcal{A}_n^\iota) \subseteq f^{M_n-1}(\mathcal{A}_{n-1}^\iota)$  and  $f^{-M_n}(\mathcal{B}_n^\iota) \subseteq f^{-M_n-1}(\mathcal{B}_{n-1}^\iota)$  (see Cor. 3.6),  $(\Phi/\Psi)_n$  implies

$$\begin{aligned} J_n^\varphi(a_n(\omega) + \omega) \cap J_n^\psi(a_n(\omega) + \omega) &= \emptyset , \\ J_n^\varphi(b_n(\omega) + \omega) \cap J_n^\psi(b_n(\omega) + \omega) &= \emptyset . \end{aligned}$$

As  $|\partial_\theta \varphi_n^\pm - \psi_n^\pm| \geq l_n^\varphi - u_n^\psi > 0$  by assumption, this ensures that the intersection has the geometry depicted in Figure 3.1. Hence it is obvious that  $I_n^\iota$  contains exactly one connected component  $I_{n+1}^\iota$  of  $\mathcal{I}_{n+1}$ , which is not reduced to a single point. Since  $\mathcal{I}_{n+1}$  is open by definition, this implies the first two statements of  $(\mathcal{I})_{n+1}$ . In addition  $I_{n+1}^\iota(\omega) = (a_{n+1}(\omega), b_{n+1}(\omega))$  is characterised by the equations

$$\begin{aligned}\varphi_n^+(a_{n+1}(\omega) + \omega, \omega) &= \psi_n^-(a_{n+1}(\omega) + \omega, \omega) , \\ \varphi_n^-(b_{n+1}(\omega) + \omega, \omega) &= \psi_n^+(b_{n+1}(\omega) + \omega, \omega) ,\end{aligned}$$

which yields  $(\Phi/\Psi)_{n+1}$ . Further, the estimates (3.33) and (3.34) in Lemma 3.10 imply

$$h_n^\varphi + h_n^\psi \leq \psi_n^+(a_{n+1}(\omega) + \omega, \omega) - \varphi_n^-(a_{n+1}(\omega) + \omega, \omega) \leq H_n^\varphi + H_n^\psi ,$$

and from Lemma 3.11 we obtain

$$l_n^\varphi - u_n^\psi \leq \partial_\theta(\varphi_n^- - \psi_n^+) \leq u_n^\varphi + u_n^\psi .$$

(Note that the bounds in these two lemmas do not depend on  $\omega \in \mathcal{F}_n$ .) Together, this yields (3.57).

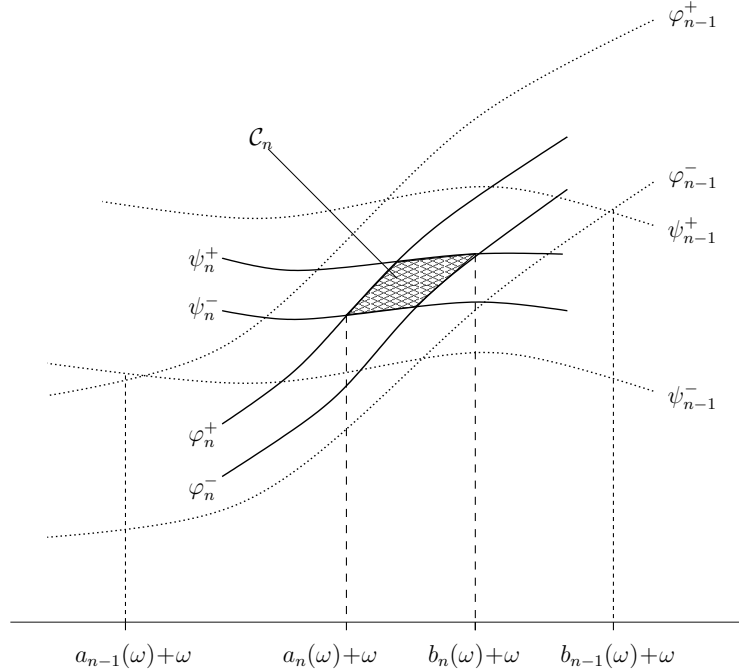


Figure 3.1: The intersection of  $f^{M_n}(\mathcal{A}_n^\iota)$  and  $f^{-M_n}(\mathcal{B}_n^\iota)$ .

In order to prove (3.58), we apply the implicit function theorem to the identity

$$\varphi_n^+(a_{n+1}(\omega) + \omega, \omega) - \psi_n^-(a_{n+1}(\omega) + \omega, \omega) = 0 ,$$

and obtain

$$\partial_\omega a_{n+1}(\omega) = \frac{(\partial_\theta + \partial_\omega)\varphi_n^+(a_{n+1}(\omega) + \omega, \omega) - (\partial_\theta + \partial_\omega)\psi_n^-(a_{n+1}(\omega) + \omega, \omega)}{\partial_\theta \varphi_n^+(a_{n+1}(\omega) + \omega, \omega) - \partial_\theta \psi_n^-(a_{n+1}(\omega) + \omega, \omega)}$$

Therefore (3.58) follows from the definitions of  $\gamma_n^\varphi, \gamma_n^\psi, l_n^\varphi$  and  $u_n^\psi$ , with the same argument applied to  $b_{n+1}$ . Consequently  $\mathcal{I}_{n+1}$  depends differentially on  $\omega \in \mathcal{F}_n$ , and the fact that the set  $\mathcal{F}_{n+1}$  is open follows quite easily from its definition. Thus  $(\mathcal{I})_{n+1}$  (iii) holds as well, and this completes the proof.

□

We summarise the results of this section in the following proposition, which is already adapted for its use in the later sections. This is also the reason why we make the dependence of  $\mathcal{F}_n$  on  $M_0, \dots, M_n$  explicit in the statement.

**Proposition 3.14.** *Suppose (A1)–(A7) hold and let  $\omega \in \mathcal{F}_n(M_0, \dots, M_n)$ . Further, assume that*

$$(3.59) \quad \mathcal{S} := s - S \cdot \left( \frac{1}{\alpha_-^{-1} - 1} + \frac{1}{\alpha_+ - 1} \right) \geq \frac{s}{2}$$

and

$$(3.60) \quad \gamma := S \cdot \sum_{k=1}^{\infty} (k\alpha_-^k + (k+1)\alpha_+^{-k}) \leq \frac{\mathcal{S}}{4}.$$

Then  $(\mathcal{I})_{n+1}$  and  $(\Phi/\Psi)_{n+1}$  hold and for all  $j = 1, \dots, n+1$  and  $\iota = 1, \dots, \mathcal{N}$  we have

$$(3.61) \quad |I_j^\iota| \leq \frac{2}{s} \cdot \max\{\alpha_-, \alpha_+^{-1}\}^{M_{j-1}},$$

$$(3.62) \quad |\partial_\omega I_j^\iota| \leq \frac{1}{4}.$$

*Proof.* Suppose that (A1)–(A7) hold. As already mentioned before,  $(\mathcal{I})_0$  and  $(\Phi/\Psi)_0$  follow directly from (A1) and the definition of  $\mathcal{F}_0$ . We proceed by induction.

Assume that  $(\mathcal{I})_n$  and  $(\Phi/\Psi)_n$  hold for some  $n \geq 0$  and  $\omega \in \mathcal{F}_n$ . Due to Lemma 3.11 and (3.59) we have  $l_n^\varphi - u_n^\psi \geq \mathcal{S} \geq s/2 > 0$ . Therefore we can apply Lemma 3.13, which implies that  $(\mathcal{I})_{n+1}$ ,  $(\Phi/\Psi)_{n+1}$  hold. Hence, the required estimates on  $|I_j^\iota|$  and  $|\partial_\omega I_j^\iota|$  follow from Lemma 3.13, in combination with Lemma 3.10, Lemma 3.12 and the estimates provided by (3.59) and (3.60). □

### 3.4 Good frequencies

In order to prove Theorem 2.1, we will have to show that under the hypothesis of the theorem there exists a set  $\Omega \subseteq \mathbb{T}^1$  of positive measure with the property that for any  $\omega \in \Omega$  one can find a monotonically increasing sequence  $(M_n(\omega))_{n \in \mathbb{N}_0}$  of positive integers, such that

$$\omega \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_n(M_0(\omega), \dots, M_n(\omega)).$$

The problem is that in order to choose the sequences  $M_n(\omega)$  inductively for a sufficiently large set of  $\omega$ , we will have to make use of the estimates on the length of the connected components of  $\mathcal{I}_n$  in Proposition 3.14. However, these estimates depend in turn on the choice of the sequence  $(M_n(\omega))_{n \in \mathbb{N}_0}$ . In order to overcome this obstacle, we restrict ourselves to choosing the sequences  $(M_n(\omega))_{n \in \mathbb{N}_0}$  from the set

$$\mathcal{M} := \{(M_n)_{n \in \mathbb{N}_0} \mid M_n \in [N_n, 2N_n] \ \forall n \in \mathbb{N}_0\},$$

where  $(N_n)_{n \in \mathbb{N}_0}$  is a sequence of positive numbers which is fixed *a priori* (for simplicity, we do not assume that the  $N_n$  are integers). In this way we can verify that all required estimates hold, independent of the particular choice of  $(M_n(\omega))_{n \in \mathbb{N}_0}$  in  $\mathcal{M}$ .

We remark that the results of this section are completely independent of the preceding one. In fact, they do not even involve the dynamics of the system. We only assume that  $(\mathcal{I}_n)_{n \in \mathbb{N}_0}$  is a family of subsets of  $\mathbb{T}^1$ , such that  $\mathcal{I}_n$  depends on the integers  $M_0, \dots, M_{n-1}$  and on  $\omega$  (as before, we keep this dependence implicit). While we will

make use of the notation introduced in Definition 3.2, we do not use the fact that the sets  $\mathcal{I}_n$  are defined dynamically as in Definition 3.1 .

As an inductive assumption, we will suppose that for given sequences  $(K_n)_{n \in \mathbb{N}_0}$  and  $(\varepsilon_n)_{n \in \mathbb{N}_0}$  in Definition 3.2 and a monotonically increasing sequence  $(N_n)_{n \in \mathbb{N}_0}$  of integers with  $N_0 \geq 2$  the following holds:

If  $n \in \mathbb{N}_0$ ,  $M_j \in [N_j, 2N_j) \forall j \in [1, n]$  and  $\omega \in \mathcal{F}_n(M_0, \dots, M_n)$ , then

$$\begin{aligned}
 (i) \quad & (\mathcal{I}1)_{n+1} \text{ holds,} \\
 (ii) \quad & |I_j^\iota| \leq \varepsilon_j \quad \forall j \in [0, n+1], \iota \in [1, \mathcal{N}] \\
 (iii) \quad & |\partial_\omega I_j^\iota| \leq \frac{1}{4} \quad \forall j \in [0, n+1], \iota \in [1, \mathcal{N}]
 \end{aligned}$$

Finally, we assume that

$$(\mathcal{N}2) \quad N_0 \geq 3 \quad \text{and} \quad N_{n+1} > 2K_n N_n \quad \forall n \in \mathbb{N}_0 .$$

**Lemma 3.15.** *Suppose  $(\mathcal{N}1)$  and  $(\mathcal{N}2)$  hold and let  $M_j \in [N_j, 2N_j)$  be fixed for  $j \in [0, n]$ . Further assume that*

$$(\mathcal{K}) \quad \sum_{j=0}^{\infty} \frac{1}{K_j} < \frac{1}{6\mathcal{N}^2} .$$

*Then for every  $\omega \in \mathcal{F}_n(M_0, \dots, M_n)$  there exists an integer  $M \in [N_{n+1}, 2N_{n+1})$  such that*

$$d((\mathcal{I}_{n+1} - (M-1)\omega) \cup (\mathcal{I}_{n+1} + (M+1)\omega), \mathcal{Y}_n) > \varepsilon_n .$$

*Proof.* If  $j \in [0, n]$  then  $\mathcal{I}_{n+1} \subseteq \mathcal{I}_j$  and  $\varepsilon_n \leq \varepsilon_j$ . Therefore

$$(3.63) \quad d\left(\mathcal{I}_{n+1} - (p-1)\omega, \bigcup_{k=-M_j+1}^{M_j+1} \mathcal{I}_j + k\omega\right) \leq \varepsilon_n$$

implies

$$(3.64) \quad d\left(\mathcal{I}_j - (p-1)\omega, \bigcup_{k=-M_j+1}^{M_j+1} \mathcal{I}_j + k\omega\right) \leq \varepsilon_j .$$

We are going to estimate the number of integers in  $(N_{n+1}, N_{n+1} + 2K_n M_n] \subseteq [N_{n+1}, 2N_{n+1})$  for which (3.64) can happen. Due to  $(\mathcal{F}1)_n$  and  $(\mathcal{N}1)(ii)$ , for any  $j \in [0, n]$ ,  $\iota, \kappa \in [1, \mathcal{N}]$  and any interval  $J \subseteq \mathbb{Z}$  of length  $|J| \leq 2K_j M_j$ , there is at most one  $p \in J$  such that  $d(I_j^\iota - (p-1)\omega, I_j^\kappa) \leq \varepsilon_j$ . Hence, there are at most  $2M_j + 1$  integers  $p$  in  $J$  such that

$$(3.65) \quad d\left(I_j^\iota - (p-1)\omega, \bigcup_{k=-M_j+1}^{M_j+1} I_j^\kappa + k\omega\right) \leq \varepsilon_j ,$$

and consequently, due to  $(\mathcal{I})_n(i)$ , at most  $\mathcal{N}^2(2M_j + 1)$  integers  $p$  in  $J$  such that

$$(3.66) \quad d\left(\mathcal{I}_j - (p-1)\omega, \bigcup_{k=-M_j+1}^{M_j+1} \mathcal{I}_j + k\omega\right) \leq \varepsilon_j .$$

Dividing the interval  $(N_{n+1}, N_{n+1} + 2K_n M_n]$  into subintervals of length  $2K_j M_j$ , plus maybe one shorter, we obtain that the number of  $p$  in  $(N_{n+1}, N_{n+1} + 2K_n M_n]$  for which (3.64) holds is bounded by

$$\left(\frac{K_n M_n}{K_j M_j} + 1\right) \mathcal{N}^2(2M_j + 1) \leq \frac{6K_n M_n \mathcal{N}^2}{K_j} .$$

Summing up over all  $j$ , this yields that there are at most

$$2K_n M_n \cdot 3\mathcal{N}^2 \cdot \sum_{j=0}^n \frac{1}{K_j}$$

$p$  in  $(N_{n+1}, N_{n+1} + 2K_n M_n]$  with  $d(\mathcal{I}_{n+1} - (p-1)\omega, \mathcal{Y}_n) \leq \varepsilon_n$ . Repeating this argument yields the same bound for the number of  $p$  in  $(N_{n+1}, N_{n+1} + K_n M_n]$  with  $d(\mathcal{I}_{n+1} + (p+1)\omega, \mathcal{Y}_n) \leq \varepsilon_n$ . Hence, due to  $(\mathcal{K})$  there must be at least one integer  $M \in (N_{n+1}, N_{n+1} + K_n M_n] \subseteq (N_{n+1}, 2N_{n+1}]$  with the required property.  $\square$

The following lemma is taken from [19]:

**Lemma 3.16.** *Suppose  $\mathcal{I} = \mathcal{I}(\omega)$  consists of exactly  $\mathcal{N}$  connected components  $I^1, \dots, I^{\mathcal{N}}$ , each of length  $|I^i| \leq \delta$  and satisfying  $|\partial_\omega I^i| \leq \gamma < \frac{1}{2}$ . Then for  $M \geq 2$  and  $\varepsilon > 0$  the set*

$$\left\{ \omega \in \mathbb{T}^1 \left| d \left( \mathcal{I}, \bigcup_{j=1}^M \mathcal{I} + j\omega \right) < \varepsilon \right. \right\}$$

*has measure  $\leq 2\mathcal{N}^2 M \frac{\delta + \varepsilon}{1 - 2\gamma}$  and consists of at most  $\mathcal{N}^2 M^2 - 1$  connected components.*

For any  $n \in \mathbb{N}_0$  let

$$(3.67) \quad u_{n+1} := 64 \cdot \mathcal{N}^2 \cdot K_{n+1} \cdot N_{n+1}^2 \cdot \frac{\varepsilon_{n+1}}{\varepsilon_n}$$

$$(3.68) \quad v_{n+1} := \frac{8}{\varepsilon_n} \cdot \mathcal{N}^2 \cdot K_{n+1}^2 \cdot N_{n+1}^3$$

Further, let  $u_0 := 32\mathcal{N}^2 K_0 N_0 \varepsilon_0$  and  $v_0 := 4\mathcal{N}^2 K_0^2 N_0^2$ .

**Lemma 3.17.** *Suppose  $(\mathcal{N}1)$ ,  $(\mathcal{N}2)$  and  $(\mathcal{K})$  hold and  $n \geq 0$ . Let  $M_j \in [N_j, 2N_j]$  be fixed for  $j \in [0, n]$  and assume  $\Lambda \subseteq \mathcal{F}_n(M_0, \dots, M_n)$  is an interval. Then for some  $r \leq v_{n+1}$  and  $\nu = 1, \dots, r$  there exist disjoint intervals  $\Lambda^\nu \subseteq \Lambda$  and numbers  $M^\nu \in [N_{n+1}, 2N_{n+1})$  such that*

$$(3.69) \quad \Lambda^\nu \subseteq \mathcal{F}_{n+1}(M_0, \dots, M_n, M^\nu)$$

and

$$(3.70) \quad \sum_{\nu=1}^r \text{Leb}(\Lambda^\nu) \geq \text{Leb}(\Lambda) - u_{n+1}.$$

*Proof.* Obviously  $\Lambda$  can be divided into at most  $\frac{2N_{n+1}}{\varepsilon_n}$  intervals  $\Gamma^\kappa$  of length  $\leq \frac{2\varepsilon_n}{3N_{n+1}}$ . For each  $\kappa$ , let  $\omega^\kappa$  be the midpoint of  $\Gamma^\kappa$ . According to Lemma 3.15, there exist integers  $M^\kappa \in [N_{n+1}, 2N_{n+1})$ , such that

$$d((\mathcal{I}_{n+1} - (M^\kappa - 1)\omega^\kappa) \cup (\mathcal{I}_{n+1} + (M^\kappa + 1)\omega^\kappa), \mathcal{Y}_n) > \varepsilon_n.$$

As  $M_j \leq 2N_j < N_{n+1} \ \forall j \in [0, n]$  and  $|\partial_\omega I_j^k| \leq \gamma \leq \frac{1}{4} \ \forall k, j$  we obtain

$$d((\mathcal{I}_{n+1} - (M^\kappa - 1)\omega) \cup (\mathcal{I}_{n+1} + (M^\kappa + 1)\omega), \mathcal{Y}_n) > 0 \quad \forall \omega \in \Gamma^\kappa.$$

Thus  $(\mathcal{F}2)_{n+1}$  holds for all  $\omega \in \Gamma^\kappa$ .

Let  $\tilde{\Gamma}^\kappa$  be the set of those  $\omega$ 's in  $\Gamma^\kappa$  that satisfy  $(\mathcal{F}1)_{n+1}$ . We have to estimate the size and the number of connected components of  $\tilde{\Gamma}^\kappa$ . However, since it follows from  $(\mathcal{N}1)(i)$  and  $(ii)$  that  $\mathcal{I}_{n+1}$  consists of  $\mathcal{N}$  connected components of length  $\leq \varepsilon_{n+1}$  and

$|\partial_\omega I_{n+1}^\nu| \leq \frac{1}{4} \forall \nu \in [1, \mathcal{N}]$  by  $(\mathcal{N}1)(iii)$ , Lemma 3.16 with  $\delta = \varepsilon_{n+1}$ ,  $\varepsilon = 3\varepsilon_{n+1}$ ,  $\gamma = \frac{1}{4}$  (see  $(\mathcal{N}1)(iii)$ ) and  $M = 2K_{n+1}N_{n+1}$  yields

$$\text{Leb}(\tilde{\Gamma}^\kappa \setminus \Gamma^\kappa) \leq 32\mathcal{N}^2 K_{n+1} N_{n+1} \varepsilon_{n+1} ,$$

and the number of connected components of  $\Gamma^\kappa$  is at most  $4\mathcal{N}^2 K_{n+1}^2 N_{n+1}^2$ . Summing up over  $\kappa$  yields the statement.  $\square$

Let

$$V_{-1} := 1 \quad \text{and} \quad V_n := \prod_{i=0}^n v_i \quad \forall n \geq 0 .$$

**Proposition 3.18.** *Suppose  $(\mathcal{N}1)$ ,  $(\mathcal{N}2)$  and  $(\mathcal{K})$  hold and*

$$(3.71) \quad \sigma := 1 - \sum_{n=0}^{\infty} V_{n-1} u_n .$$

*Then there exists a set  $\Omega \subseteq \mathbb{T}^1$  of measure  $\text{Leb}(\Omega) \geq \sigma$ , such that for each  $\omega \in \Omega$  there exists a sequence  $(M_n(\omega))_{n \in \mathbb{N}_0}$  with the property that*

$$(3.72) \quad \omega \in \bigcap_{n \in \mathbb{N}_0} \mathcal{F}_n(M_0(\omega), \dots, M_n(\omega)) .$$

*Proof.* We are going to construct a nested sequence of sets  $\mathbb{T}^1 \supseteq \Omega_0 \supseteq \Omega_1 \supseteq \dots$  with the following properties:

- (i)  $\Omega_n$  consists of  $\rho_n \leq V_n$  disjoint open intervals  $\Omega_n^1, \dots, \Omega_n^{\rho_n}$ .
- (ii)  $\text{Leb}(\Omega_n) \geq 1 - \sum_{i=0}^n V_{i-1} u_i$
- (iii) For each  $i = 1, \dots, \rho_n$  there exist numbers  $M_0^{n,i}, \dots, M_n^{n,i}$  such that

$$\Omega_n^i \subseteq \mathcal{F}_n(M_0^{n,i}, \dots, M_n^{n,i}) .$$

- (iv) For each  $k \leq n$  and each  $i \in [1, \rho_n]$  there exists a unique  $\kappa \in [1, \rho_k]$  such that  $\Omega_n^i \subseteq \Omega_k^\kappa$  and  $M_j^{n,i} = M_j^{k,\kappa} \forall j = 0, \dots, k$ .

For  $n = 0$  we choose  $\Omega_0 = \mathcal{F}_0$ . Recall that this is the set of all  $\omega$  which satisfy condition  $(\mathcal{F}1)_0$ , and the fact that this set has all required properties can be deduced from Lemma 3.16 .

Now suppose  $\Omega_0, \dots, \Omega_n$  with the above properties exists. Then for each  $i \in [1, \rho_n]$  we can apply Lemma 3.17 to the component  $\Omega_n^i$  and obtain a union of at most  $v_{n+1}$  intervals with overall measure  $\geq m(\Omega_n^i) - u_{n+1}$ . Doing this for all the at most  $V_n$  components of  $\Omega_n$  yields the required set  $\Omega_{n+1}$ , with at most  $V_{n+1} = v_{n+1} \cdot V_n$  connected components and measure  $\geq 1 - \sum_{i=0}^{n+1} V_{i-1} u_i$ .

As the sets  $\Omega_n$  form a nested sequence, their intersection  $\Omega$  has measure  $\geq \sigma$ . Further, for any  $\omega \in \Omega$  and  $n \in \mathbb{N}$  there exists a unique  $i_n \in [1, \rho_n]$  with  $\omega \in \Omega_n^{i_n}$ . If we let  $M_n(\omega) = M_n^{n,i_n}$ , then due to property (iv) we obtain (3.72).  $\square$

### 3.5 Proof of Theorem 2.1, Part A: Existence of SNA

Suppose that the assumptions of Theorem 2.1 hold. First of all, we choose the sequence  $K_n$  in a way that allows to obtain a lower bound on the asymptotic expansion and contraction rate, namely

$$(3.73) \quad \min\{\alpha_-^{-1}, \alpha_+\} \geq \alpha^{\frac{1}{p}} .$$

In order to do so, we fix  $t \in \mathbb{N}$  sufficiently large, such that  $t \geq 4$  and

$$\frac{2^{-t+2}}{\mathcal{N}^2} \leq \log \left( \frac{p^2+2}{p^2+1} \right) .$$

Then we let  $K_n := 2^{n+t} \mathcal{N}^2$ . Note that this choice satisfies  $(\mathcal{K})$ . We obtain

$$\beta = \prod_{n=0}^{\infty} \left( 1 - \frac{1}{K_n} \right) \geq \exp \left( -2 \sum_{n=0}^{\infty} \frac{1}{K_n} \right) \geq \frac{p^2+1}{p^2+2} ,$$

and this implies

$$\alpha_-^{-1} = \alpha^{\frac{2}{p}\beta - p(1-\beta)} \geq \alpha^{\frac{1}{p}} .$$

Similarly we obtain  $\alpha_+ \geq \alpha^{\frac{1}{p}}$ , such that (3.73) holds.

Now let  $N_0 := 3$  and  $N_{n+1} := \alpha^{N_n/16p}$ . As the sequence  $N_n$  grows super-exponentially,  $(\mathcal{N}2)$  holds whenever  $\alpha$  is sufficiently large. Further, let

$$\varepsilon_0 := \min_{i=1}^{\mathcal{N}} |I_0^i| \quad \text{and} \quad \varepsilon_n := \frac{2}{s} \cdot \alpha^{-N_{n-1}/p} .$$

Again, if  $\alpha$  is sufficiently large, then on the one hand  $\varepsilon_n \geq 3\varepsilon_{n+1} \forall n \in \mathbb{N}_0$  (which is the only requirement on the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  in Definition 3.2), and on the other hand (3.59) and (3.60) hold. Therefore we can apply Proposition 3.14 to see that  $(\mathcal{N}1)$  holds for the sets  $\mathcal{I}_n$  given by Definition 3.1. This means that all assumptions of Proposition 3.17 are met, and we obtain a set  $\Omega \subseteq \mathbb{T}^1$  of measure

$$(3.74) \quad \text{Leb}(\Omega) \geq 1 - \sum_{n=0}^{\infty} V_{n-1} u_n ,$$

with the property that for all  $\omega \in \Omega$  there exists a sequence  $(M_n(\omega))_{n \in \mathbb{N}}$ , such that  $\omega \in \bigcap_{n \in \mathbb{N}_0} \mathcal{F}_n(M_0(\omega) \dots M_n(\omega))$ . Proposition 3.9 then implies that for all  $\omega \in \Omega$  the system

$$f(\theta, x) = (\theta + \omega, f_{\theta}(x))$$

has a sink-source-orbit, and consequently a SNA and SNR by Proposition 1.1. It remains to estimate the size of  $\Omega$ , i.e. to obtain a lower bound on the right side of (3.74).

In all of the following estimates we assume that  $\alpha$  is chosen sufficiently large, such that in particular the sequence  $N_n$  grows sufficiently fast, and indicate the steps in which this fact is used by placing  $(\alpha)$  over the respective inequality signs. For any  $n \in \mathbb{N}_0$  we have

$$\begin{aligned} u_{n+1} &= 64\mathcal{N}^2 \cdot K_{n+1} \cdot N_{n+1}^2 \cdot \frac{\varepsilon_{n+1}}{\varepsilon_n} \\ &= 64\mathcal{N}^2 \cdot K_{n+1} \cdot \alpha^{N_n/8p - N_n/p + N_{n-1}/p} \stackrel{(\alpha)}{\leq} \alpha^{-3N_n/4p} \end{aligned}$$

and

$$\begin{aligned} v_{n+1} &= \frac{8}{\varepsilon_n} \cdot \mathcal{N}^2 \cdot K_{n+1}^2 \cdot N_{n+1}^3 \\ &\leq 4s \cdot \mathcal{N}^2 \cdot K_{n+1}^2 \cdot \alpha^{N_{n-1}/p + 3N_n/16p} \stackrel{(\alpha)}{\leq} \alpha^{N_n/4p} . \end{aligned}$$

Now note that

$$V_0 = v_0 = \mathcal{N}^2 \cdot K_0^2 \cdot N_0^2 \stackrel{(\alpha)}{\leq} \alpha^{N_0/4p} .$$



Further, if we suppose that

$$(3.75) \quad V_n \leq \alpha^{N_n/4p}$$

then

$$V_{n+1} = V_n \cdot v_{n+1} \leq \alpha^{N_n/4p+N_n/4p} \stackrel{(\alpha)}{\leq} \alpha^{N_{n+1}/4p}.$$

Consequently, by induction, (3.75) holds for all  $n \geq 1$ . We conclude

$$V_n u_{n+1} \leq \alpha^{-N_n/4p}$$

and

$$(3.76) \quad 1 - \sum_{n=0}^{\infty} V_{n-1} u_n \geq 1 - u_0 - \sum_{n=0}^{\infty} \alpha^{-N_n/4p}.$$

As  $u_0 = 32\mathcal{N}^2 K_0 N_0 \varepsilon_0 \rightarrow 0$  if  $\varepsilon_0 \rightarrow 0$ , the right side is arbitrarily close to 1 if  $\alpha$  is large and  $\varepsilon_0$  is small.

To summarise, this means that we can choose constants  $\tilde{\alpha}_0$  and  $\tilde{c}_0$  in such a way that all the assumptions on  $\alpha$  used above hold and (3.76) is larger than  $1 - \delta$  whenever  $\alpha \geq \tilde{\alpha}_0$  and  $\varepsilon_0 \leq \tilde{c}_0$ . Then  $\text{Leb}(\Omega) \geq 1 - \delta$ , as required. This proves Theorem 2.1, except for the minimality.

### 3.6 Proof of Theorem 2.1, Part B: Minimality

We choose  $\tilde{\alpha}_0$  and  $\tilde{c}_0$  as at the end of the preceding section and suppose  $\alpha \geq \tilde{\alpha}_0$  and  $\varepsilon_0 \leq \tilde{c}_0$ . Further, we fix  $\omega \in \Omega$  and the corresponding sequence  $(M_n)_{n \in \mathbb{N}} = (M_n(\omega))_{n \in \mathbb{N}}$  and let  $f(\theta, x) = (\theta + \omega, f_\theta(x))$  as before. Recall that  $M_n \in [N_n, 2N_n)$  and  $N_{n+1} = \alpha^{N_n/16p}$ .

We start with some preliminary remarks and estimates. Let  $\varepsilon_n$  and  $N_n$  be chosen as in the last section. Since  $\alpha \geq \tilde{\alpha}_0$  and  $\omega \in \mathcal{F}_n \forall n \in \mathbb{N}_0$ , the assumptions of Proposition 3.14 are satisfied for all  $n \in \mathbb{N}_0$ . Consequently, for all  $n \geq 0$  the statements  $(\mathcal{I})_n$  and  $(\Phi/\Psi)_n$  hold and

$$|I_n^\iota| \leq \varepsilon_n \quad \forall \iota \in [1, \mathcal{N}].$$

Let

$$\Theta := \mathbb{T}^1 \setminus \bigcup_{n \in \mathbb{N}_0} \mathcal{Z}_n.$$

Then

$$\begin{aligned} \text{Leb}(\Theta) &\geq 1 - \sum_{n=0}^{\infty} m(\mathcal{Z}_n) \geq 1 - \sum_{n=0}^{\infty} 4N_n \varepsilon_n \\ &= 1 - N_0 \varepsilon_0 - \sum_{n=1}^{\infty} \frac{8}{s} \cdot \alpha^{N_{n-1}/16p - N_{n-1}/p}. \end{aligned}$$

We now choose the constants  $\alpha_0 \geq \tilde{\alpha}_0$  and  $c_0 \leq \tilde{c}_0$ , such that for all  $\alpha \geq \alpha_0$  and  $\varepsilon_0 \leq c_0$  there holds

$$(3.77) \quad \text{Leb}(\Theta) > 1 - \frac{1}{4(1+p^2)}.$$

Let

$$S^* := S + \frac{S}{\alpha_-^{-1} - 1}$$

and choose a constant  $\Lambda > 1$  with the following property:

If  $\Gamma \subseteq \mathbb{T}^2$  is the graph of a differentiable curve  $\gamma : I \rightarrow \mathbb{T}^1$ , defined on an interval  $I \subseteq \mathbb{T}^1$ , and  $\Gamma$  has slope at most  $S^*$ , then  $f^n(\Gamma)$  has slope at most  $S^* \cdot \Lambda^n$ .

Further, due to the lower bound in (3.57) and the estimates provided by combining Lemmas 3.10, 3.11 and 3.13, there exist constants  $B > 0$  and  $\lambda > 0$ , such that for any  $n \geq 0$  and any connected component  $I_n^\iota$  of  $\mathcal{I}_n$  there holds

$$|I_n^\iota| \geq B \cdot \lambda^{-N_{n-1}}.$$

Let  $\delta_n := B \cdot \lambda^{-N_{n-1}}$ . Since

$$\text{Leb} \left( \bigcup_{k=n+1}^{\infty} \mathcal{V}_k \setminus \mathcal{V}_n \right) \leq \sum_{k=n+1}^{\infty} (M_k + 1) \cdot \varepsilon_k \leq \sum_{k=n+1}^{\infty} \frac{4}{s} \cdot \alpha^{N_{k-1}/8p - N_{k-1}/p},$$

and due to the super-exponential growth of the sequence  $N_n$ , there exists  $n_0 \geq 0$  such that for any connected component  $I_n^\iota$  of  $\mathcal{I}_n$  there holds

$$(3.78) \quad \text{Leb} \left( \bigcup_{k=n+1}^{\infty} \mathcal{V}_k \setminus \mathcal{V}_n \right) < \delta_n / 2\Lambda^{M_{n-1}} \quad \forall n \geq n_0.$$

By slightly reducing the set  $\Theta$  if necessary, it is therefore possible to find a set  $\Theta^* \subseteq \Theta$  with the following properties:

$$(\Theta^*1) \quad \text{Leb}(\Theta^*) > 1 - \frac{1}{2(1+p^2)};$$

$$(\Theta^*2) \quad \text{For any } \theta \in \Theta^*, \text{ any } n \geq n_0 \text{ and any } \iota \in [1, \mathcal{N}], \text{ the forward orbit } \{\theta + n\omega \mid n \geq 0\} \text{ is } \delta_n / \Lambda^{M_{n-1}}\text{-dense in } (I_n^\iota - (M_n - 1)\omega) \setminus \left( \bigcup_{k=n+1}^{\infty} \mathcal{V}_k \setminus \mathcal{V}_n \right).$$

Now we come to the key point of the proof. The crucial observation is the fact that there is a large set of points with dense orbit - minimality will then follow by rather general arguments. More precisely, we prove the following:

**Claim 3.19.** *Suppose  $\theta_0 \in \Theta^* \cap (\Theta^* - \omega)$  and  $x_0 \in E^c$ . Then the forward orbit of  $(\theta_0, x_0)$  is dense in  $\mathbb{T}^2$ .*

*Proof.* For any point  $(\theta_0, x_0) \in \mathbb{T}^2$ , denote its forward orbit by  $\mathcal{O}^+(\theta_0, x_0) := \{(\theta_k, x_k) \mid k \geq 0\}$ . Suppose  $\theta \in \Theta^* \cap (\Theta^* - \omega)$  and  $x \in E^c$ . Since  $\Theta^* \subseteq \Theta \subseteq \mathcal{Z}_0^c \subseteq \mathcal{I}_0^c$ , we can use (A1) to see that  $f_\theta(x) \in C$ . Therefore, it suffices to show that the forward orbit of any point  $(\theta_0, x_0)$  with  $\theta_0 \in \Theta^*$  and  $x_0 \in C$  is dense. Fix such  $\theta_0$  and  $x_0$  and any  $\iota \in [1, \mathcal{N}]$ . Further, choose  $n_0$  as in (3.78). We proceed in four steps:

Step 1: *If  $n \geq n_0$ , then  $\pi_1(\mathcal{O}^+(\theta_0, x_0) \cap \mathcal{A}_n^\iota)$  is  $\delta_n / \Lambda^{M_{n-1}}$ -dense in  $I_n^\iota - (M_n - 1)\omega$ .*

Since  $\theta_0 \in \Theta^*$ , it is not contained in  $\mathcal{Z}_n$  for any  $n \in \mathbb{N}_0$ . Hence, it follows from Lemma 3.4 that  $x_m \notin C$  implies  $\theta_m \in \mathcal{V}_k$  for some  $k \in \mathbb{N}_0$ . Now  $I_n^\iota - (M_n - 1)\omega$  is disjoint from  $\mathcal{V}_n$  by  $(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$ . Therefore  $\theta_m \in I_n^\iota - (M_n - 1)\omega$  and  $x_m \notin C$  imply  $\theta_k \in \bigcup_{k=n+1}^{\infty} \mathcal{V}_k \setminus \mathcal{V}_n$ . In other words,  $x_m \in C$  whenever  $\theta_m \in (I_n^\iota - (M_n - 1)\omega) \setminus \left( \bigcup_{k=n+1}^{\infty} \mathcal{V}_k \setminus \mathcal{V}_n \right)$ . The statement follows from property  $(\Theta^*2)$  of the set  $\Theta^*$ .

Step 2: *There exists an integer  $n_1 \geq n_0$ , such that for all  $n \geq n_1$  the set  $\pi_2(\mathcal{O}^+(\theta_0, x_0) \cap (I_{n+1}^\iota + (M_n + 1)\omega) \times \mathbb{T}^1)$  is  $2^{-n}$ -dense in  $E$ .*

Let  $n \geq n_0$ . With the notation of Section 3.3, we have

$$f^{M_{n+1}}(\mathcal{A}_{n+1}^\iota) = \{(\theta, x) \mid \theta \in I_{n+1}^\iota + \omega, x \in [\varphi_{n+1}^-(\theta), \varphi_{n+1}^+(\theta)]\}.$$

Due to the estimates (3.33) in Lemma 3.10 and (3.38) in Lemma 3.11, this set is a small strip<sup>2</sup> of vertical size at most  $\alpha^{-M_{n+1}/p}$  and slope at most  $S^*$ . As described in the proof of Lemma 3.13, this strip crosses the strip  $f^{-M_n}(\mathcal{B}_n^\ell)$  from below to above (where we assume again that the crossing is upwards), see Figure 3.1. This implies that the strip  $A := f^{M_{n+1}+M_n}(\mathcal{A}_{n+1}^\ell)$  crosses the horizontal strip  $B_n^\ell = (I_n^\ell + (M_n + 1)\omega) \times E$  in the same way.

From (A2) and  $\alpha_u = \alpha^p$ , it follows that  $A$  has vertical size at most  $\alpha^{-M_{n+1}/p+2M_np}$ . Further, it has slope at most  $S^* \cdot \Lambda^{M_n}$ . Since  $\pi_1(\mathcal{O}^+(\theta_0, x_0) \cap \mathcal{A}_{n+1}^\ell)$  is  $\delta_{n+1}/\Lambda^{M_n}$ -dense in  $I_{n+1}^\ell - (M_{n+1} - 1)\omega$  by Step 1, it follows that  $\pi_2(A)$  is  $d_n$ -dense in  $E$ , where

$$d_n = S^* \cdot \delta_{n+1} + \alpha^{-M_{n+1}/p+2M_np}.$$

Given the super-exponential growth of the sequence  $N_n$  and  $M_n$ , there exists  $n_1 \geq n_0$ , such that  $d_n \leq 2^{-n} \forall n \geq n_1$ . This completes Step 2.

Step 3:  $\text{cl}(\mathcal{O}^+(\theta_0, x_0))$  contains a vertical segment  $\{\zeta\} \times E$  for some  $\zeta \in \Theta - \omega$ .

Due to compactness and since the size of the intervals  $I_n^\ell$  goes to zero as  $n$  goes to infinity, there exists a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}}$  of integers and a point  $\zeta \in \mathbb{T}^1$ , such that the intervals  $I_{n_i+1}^\ell + (M_{n_i} + 1)\omega$  converge to  $\{\zeta\}$  in Hausdorff distance. It follows from Step 2 that  $\{\zeta\} \times E \subseteq \text{cl}(\mathcal{O}^+(\theta_0, x_0))$ .

$(\mathcal{F}1)_n$  and  $(\mathcal{F}2)_n$  imply that  $I_n^\ell + (M_n + 1)\omega$  is contained in  $\mathcal{Z}_n^c - \omega$  for all  $n \in \mathbb{N}_0$ . Since the sets  $\mathcal{Z}_n^c - \omega$  form a nested sequence of compact sets, it follows that  $\zeta$  is contained in  $\bigcap_{i=0}^\infty \mathcal{Z}_{n_i}^c - \omega = \Theta - \omega$ .

Step 4:  $\mathcal{O}^+(\theta_0, x_0)$  is dense in  $\mathbb{T}^2$ .

Let  $x^\pm := f_\zeta(e^\pm)$ . Since  $\Theta - \omega$  is disjoint from  $\mathcal{I}_0 \subseteq \mathcal{Z}_0 - \omega$ , (A1) implies  $x^\pm \in C$ . Consequently,  $(\mathcal{B}1)_n$  holds for all  $(\zeta + \omega, x)$  with  $x \in [x^+, x^-]$  and all  $n \in \mathbb{N}_0$ .

Let  $\mathcal{L}$  be the smallest positive integer such that  $\zeta + (\mathcal{L} + 1)\omega \in \mathcal{I}_n$ . Then we can use  $(\mathcal{C}3)_n$  together with (A2) and (A4) to conclude that

$$\partial_x f_{\zeta+\omega}^\mathcal{L}(x) \leq \alpha_-^\mathcal{L} \quad \forall x \in [x^+, x^-].$$

It follows that  $f^\mathcal{L}(\{\zeta + \omega\} \times [x^+, x^-])$  is a vertical segment of size smaller than  $\alpha_-^\mathcal{L}$ . Since  $\mathcal{L} \geq M_n - 1$  (due to  $\zeta + \omega \notin \mathcal{Z}_n$ ) and  $n$  was arbitrary, this means that the length of the corresponding iterates of  $\{\zeta + \omega\} \times [x^+, x^-]$  goes to zero as  $n$  goes to infinity. Therefore, the orbit of the segment  $\{\zeta + \omega\} \times [x^-, x^+] = f(\{\zeta\} \times E)$  is dense in  $\mathbb{T}^2$ . Since  $\{\zeta\} \times E \subseteq \text{cl}(\mathcal{O}^+(\theta_0, x_0))$  by Step 3, this completes the proof of Step 4 and the claim.  $\square$

The preceding claim implies in particular that  $f$  is topologically transitive. It follows from Proposition 1.3 that there is a unique minimal set  $M$ . Obviously,  $M$  cannot be a continuous invariant curve with positive Lyapunov exponent, since the complement of such a curve always contains at least one further minimal set. It follows from [20] that  $f$  must support at least one invariant measure  $\mu$  with non-positive vertical Lyapunov exponent, that is

$$(3.79) \quad \lambda(\mu) := \int_{\mathbb{T}^1} \int_{\mathbb{T}^1} \partial_x \log f_\theta(x) d\mu_\theta(x) d\theta \leq 0.$$

(Here  $\mu_\theta$  are the conditional measures with respect to the  $\sigma$ -algebra  $\pi^{-1}(\mathcal{B}(\mathbb{T}^1))$ .)

We claim that this is only possible if  $M$  intersects  $(\Theta^* \cap (\Theta^* - \omega)) \times E^c$ . In order to see this, note that due to  $(\Theta^*1)$ , the set  $(\Theta^* \cap (\Theta^* - \omega))$  has measure  $> 1 - 1/(1 + p^2)$ .

<sup>2</sup>By ‘strip’, we just mean a set which is the region between two continuous curves, defined on a subinterval of  $\mathbb{T}^1$ . By the slope of a strip we mean the slope (or derivative) of its boundary curves.

If  $\text{supp}(\mu) \subseteq M$  and  $M$  is disjoint from  $(\Theta \cap (\Theta - \omega)) \times E^c$ , it therefore follows from (A2) and (A3) that

$$\lambda(\mu) > \left(1 - \frac{1}{1+p^2}\right) \cdot \log(\alpha^{1/p}) - \frac{1}{1+p^2} \cdot \log(\alpha^p) = 0 ,$$

contradicting (3.79).

It follows that  $M$  intersects  $(\Theta^* \cap (\Theta^* - \omega)) \times E^c$ , and since all points from the later set have dense orbits by Claim 3.19 we obtain  $M = \mathbb{T}^2$ . This completes the proof of Theorem 2.1 .

### 3.7 Proof of Corollary 2.3

Obviously, we just have to check that the assumptions (A1)–(A7) of Theorem 2.1 with  $\alpha_l^{-1} = \alpha_u = \alpha^p$  are satisfied for all large  $\alpha$ . Here  $p$  is meant to be the same as in (2.2). In all of the following, we assume that  $\alpha$  is chosen sufficiently large and just indicate by  $(\alpha)$  whenever this fact is used.

Due to (2.5), there exist  $\varepsilon > 0$  and  $s > 0$ , such that

$$|g'(\theta)| > s \quad \forall \theta \in g^{-1}(B_\varepsilon(1/2)) .$$

We let  $\mathcal{I}_0 := g^{-1}(B_\varepsilon(1/2))$ , such that (A6) holds by definition. Note that due to (2.4),  $\mathcal{I}_0$  is the disjoint union of a finite number of open intervals. In addition, by reducing  $\varepsilon$  further if necessary, we can assume that all connected components have length smaller than  $\varepsilon_0$ , where  $\varepsilon_0 = \varepsilon_0(\delta, p, s, S, \mathcal{N})$  from Theorem 2.1 with  $S := \max_{\theta \in \mathbb{T}^1} |g'(\theta)|$ . Note that this choice of  $S$  automatically implies (A5).

Further, we define  $e^\pm := \pm \alpha^{-\frac{2p-1}{2p}}$  and  $c^\pm := \mp \varepsilon/2$ , and let  $E = [e^-, e^+]$  and  $C = [c^-, c^+]$  as before. Then for large  $\alpha$  we have  $h_\alpha(\mathbb{T}^1 \setminus E) \subseteq B_{\varepsilon/2}(1/2)$ , since

$$h_\alpha(e^\pm) = \pm \pi \left( \frac{a_p(\alpha^{1/2p})}{2a_p(\alpha/2)} \right) \xrightarrow{\alpha \rightarrow \infty} \frac{1}{2} .$$

Consequently  $f_\theta(\mathbb{T}^1 \setminus E) \subseteq C \quad \forall \theta \notin \mathcal{I}_0$ , such that (A1) holds. Similarly, the above choices imply that (A7) holds (provided we take  $\varepsilon < \frac{1}{2}$ ).

For any  $(\theta, x) \in \mathbb{T}^2$ , there holds

$$\begin{aligned} \partial_x f_\theta(x) &= h'_\alpha(x) \geq h'_\alpha(1/2) \\ &= \frac{\alpha \cdot a'_p(\alpha/2)}{2a_p(\alpha/2)} = \frac{\alpha \cdot (1 + (\alpha/2)^p)^{-1}}{2a_p(\alpha/2)} \stackrel{(\alpha)}{\geq} \alpha^{-p} . \end{aligned}$$

Similarly, there holds

$$\partial_x f_\theta(x) = h'_\alpha(x) \leq h'_\alpha(0) = \frac{\alpha}{a_p(\alpha/2)} \stackrel{(\alpha)}{\leq} \alpha^p .$$

Thus (A2) holds.

Finally, we check (A3) and (A4). Suppose  $x \in E$ . Then

$$\partial_x f_\theta(x) \geq h'_\alpha(x) = \frac{\alpha a'_p(\alpha e)}{2a_p(\alpha/2)} = \frac{\alpha \cdot (1 + \alpha^{1/2})^{-1}}{2a_p(\alpha/2)} \stackrel{(\alpha)}{\geq} \alpha^{1/p} .$$

Similarly, if  $x \in C$  there holds

$$\partial_x f_\theta(x) \leq h'_\alpha(\varepsilon) = \frac{\alpha \cdot (1 + (\alpha \varepsilon)^p)^{-1}}{2a_p(\alpha/2)} \stackrel{(\alpha)}{\leq} \alpha^{-1/p} .$$

It follows that for sufficiently large  $\alpha$  all the assumptions of Theorem 2.1 are satisfied. This completes the proof of the corollary.

## 4 Proof of the refined statement for the qpf Arnold circle map

### 4.1 Proof of Theorem 2.5

In this section, we describe how the basic construction has to be modified in order to prove Theorem 2.5. In fact, only minor changes are needed. The only thing which has to be done is to improve some of the estimates in Section 3.3, taking advantage of the additional assumption (A8), and then adapt the proof from Section 3.5 accordingly. We remark that all results of Sections 3.2 and 3.3 only depend on the assumptions (A1)–(A7) and not on the fact that the parameter  $\alpha$  is chosen very large. Therefore, they all apply in the situation of Theorem 2.5. Similarly, we can still use all results of Section 3.4, since these were completely independent of the dynamics.

First of all, we slightly modify the definition of the sets  $\mathcal{F}_n$ : We replace condition  $(\mathcal{F}1)_0$  by

$$(\mathcal{F}1')_0 \quad d\left(\mathcal{I}'_0, \bigcup_{k=1}^{2K_0M_0} (\mathcal{I}'_0 + k\omega)\right) > 3\varepsilon_0$$

and define  $\mathcal{F}'_n$  as the set of all frequencies  $\omega \in \mathbb{T}^1$  which satisfy  $(\mathcal{F}1')_0$ ,  $(\mathcal{F}2)_0$  and  $(\mathcal{F}1-2)_j \forall j = 1, \dots, n$ . Since  $\mathcal{I}_0 \subseteq \mathcal{I}'_0$ , condition  $(\mathcal{F}1')_0$  is stronger than  $(\mathcal{F}1)_0$ , which means that  $\mathcal{F}'_n \subseteq \mathcal{F}_n$ . Consequently, all the results from Sections 3.2–3.4 remain true if  $\mathcal{F}_n$  is replaced by  $\mathcal{F}'_n$  in the respective statements.

Since the expansion and contraction rates in Theorem 2.5 are fixed, we have to improve the estimates from Section 3.3, making use of the strengthened condition  $(\mathcal{F}1')_0$  together with the additional assumption (A8). As the proofs are just slight variations of the corresponding ones in Section 3.3, we keep the exposition rather brief and only describe the needed modifications. First of all, Lemma 3.11 will be replaced by the following:

**Lemma 4.1.** *Suppose (A1)–(A8) hold and  $\omega \in \mathcal{F}'_n$ . Then*

$$(4.1) \quad s - \frac{s' + \alpha_-^{M_0} S}{\alpha_-^{-1} - 1} \leq l_n^\varphi \leq u_n^\varphi \leq S + \frac{s' + \alpha_-^{M_0} S}{\alpha_-^{-1} - 1}$$

and

$$(4.2) \quad u_n^\psi \leq \frac{s' + \alpha_+^{M_0} S}{\alpha_+ - 1}.$$

*Proof.* As in the proof of Lemma 3.11, we fix  $\theta \in I_n^\iota + \omega$  and first let  $(\theta_0, x_0) = (\theta - M_n\omega, c^\pm)$  and  $\mathcal{L} = M_n - 1$ , such that  $f_{\theta_0}^{\mathcal{L}+1}(x_0) = \varphi_n^\pm(\theta)$ . We obtain

$$\begin{aligned} \partial_\theta \varphi_n^\pm(\theta) &= \partial_\theta f_{\theta_0}^{\mathcal{L}+1}(x_0) \\ (3.41) \quad &= \partial_\theta f_{\theta_\mathcal{L}}(x_\mathcal{L}) + \sum_{k=0}^{\mathcal{L}-1} \partial_x f_{\theta_{k+1}}^{\mathcal{L}-k}(x_{k+1}) \cdot \partial_\theta f_{\theta_k}(x_k) \\ (\mathcal{F}1')_0 \quad &\geq s - \sum_{k=\mathcal{L}-M_0}^{\mathcal{L}-1} \alpha_-^{\mathcal{L}-k} s' - \sum_{k=0}^{\mathcal{L}-M_0-1} \alpha_-^{\mathcal{L}-k} S \\ &= s - \frac{s' + \alpha_-^{M_0} S}{\alpha_-^{-1} - 1}. \end{aligned}$$

The second estimate in (4.1) follows in the same way.

In order to prove (4.2), we can proceed similarly: We let  $(\theta_0, x_0) = (\theta + M_n \omega, e^\pm)$  and  $\mathcal{R} = M_n$ , such that  $f_{\theta_0}^{-\mathcal{R}}(x_0) = \psi_n^\pm(\theta)$ , and obtain the required estimate from (3.43) and (3.45) by using  $(\mathcal{F}1')_0$  once more.  $\square$

Next, we derive an improved version of Lemma 3.12:

**Lemma 4.2.** *Suppose (A1)–(A8) hold and  $\omega \in \mathcal{F}'_n$ . Then*

$$(4.3) \quad \gamma_n^\varphi \leq s' \cdot \sum_{k=1}^{\infty} k \alpha_-^k + S \cdot \sum_{k=M_0+1}^{\infty} k \alpha_-^k$$

and

$$(4.4) \quad \gamma_n^\psi \leq s' \cdot \sum_{k=1}^{\infty} (k+1) \alpha_+^{-k} + S \cdot \sum_{k=M_0+1}^{\infty} (k+1) \alpha_+^{-k}.$$

*Proof.* The proof is almost identical to that of Lemma 3.12. For proving the upper bound on  $\gamma_n^\varphi$ , the only difference is that (A8) is used instead of (A5) in order to estimate  $|\partial_\theta f_{\theta_k}(x_k)|$  in the last  $M_0$  terms of the sum in (3.54).

Similarly, the improved bound on  $\gamma_n^\psi$  is obtained by using (A8) instead of (A5) when the last  $M_0$  terms of the sum on the right side of (3.56) are estimated via (3.45).  $\square$

Lemma 3.13 can be used without any modifications. Consequently, we arrive at the following conclusion, whose proof is identical to that of Proposition 3.14.

**Proposition 4.3.** *Suppose (A1)–(A8) hold and let  $\omega \in \mathcal{F}'_n(M_0, \dots, M_n)$ . Further, assume that*

$$(4.5) \quad \mathcal{S}' := s - \left( \frac{s' + \alpha_-^{M_0} S}{\alpha_-^{-1} - 1} + \frac{s' + \alpha_+^{-M_0} S}{\alpha_+ - 1} \right) \geq \frac{s}{2}$$

and

$$(4.6) \quad \gamma' := s' \cdot \sum_{k=1}^{\infty} (k \alpha_-^k + (k+1) \alpha_+^{-k}) + S \cdot \sum_{k=M_0+1}^{\infty} (k \alpha_-^k + (k+1) \alpha_+^{-k}) \leq \frac{\mathcal{S}'}{4}.$$

Then  $(\mathcal{I})_{n+1}$  and  $(\Phi/\Psi)_{n+1}$  hold and for all  $j = 1, \dots, n+1$  and  $\iota = 1, \dots, \mathcal{N}$  we have

$$(4.7) \quad |I_j^\iota| \leq \frac{2}{s} \cdot \max\{\alpha_-, \alpha_+^{-1}\}^{M_{j-1}},$$

$$(4.8) \quad |\partial_\omega I_j^\iota| \leq \frac{1}{4}.$$

In order to complete the proof of Theorem 2.5, we now choose the sequence  $(K_n)_{n \in \mathbb{N}_0}$  as in the proof of Theorem 2.1, such that  $\alpha_-^{-1}, \alpha_+ \geq \alpha^{1/p}$ . Further, we let  $N_0$  be the smallest integer larger than  $d^{1/4}$ . In all of the following, we assume that  $d$  is chosen sufficiently large to ensure all the required estimates. As before, we define the sequence  $(N_n)_{n \in \mathbb{N}}$  recursively by  $N_{n+1} = \alpha^{N_n/16p}$  and let

$$\varepsilon_0 := \min_{\iota=1}^{\mathcal{N}} |I_0^\iota| \quad \text{and} \quad \varepsilon_n := \frac{2}{s} \cdot \alpha^{-N_{n-1}/p}.$$

If  $d_0$  (and consequently  $N_0$ ) is chosen large enough, then  $(\mathcal{N}2)$  holds and  $\varepsilon_n \geq 3\varepsilon_{n+1} \forall n \in \mathbb{N}$ . Further, (4.5) and (4.6) hold if  $d_0$  is large and  $s'/s$  is small (note that the product

$\alpha^{-M_0} S \leq \alpha^{-N_0} S$  decays super-exponentially as  $d$  is increased). Thus  $(\mathcal{N}1)$  holds by Proposition 4.3 . Therefore, we can apply Proposition 3.17 and obtain

$$\text{Leb}(\Omega) \geq 1 - \sum_{n=0}^{\infty} V_{n-1} u_n .$$

From now on the proof is identical to the one of Theorem 2.1, with the only difference that the largeness condition on  $\alpha$  is replaced by a largeness condition on  $d$  (and thus  $N_0$ ) in all the respective estimates. In this way, we obtain

$$\text{Leb}(\Omega) \geq 1 - u_0 - \sum_{n=0}^{\infty} \alpha^{-N_n/4p} .$$

If  $d$  goes to infinity, then due to (2.9) and the choice of  $N_0$  the right side tends to 1 (recall that  $u_0 = 32\mathcal{N}^2 K_0 N_0 \varepsilon_0$ ).

The proof of minimality given in Section 3.6 literally stays the same. The only thing which has to be noted is that the estimate in (3.77) also holds for fixed  $\alpha$ , provided  $N_0 \approx d^{1/4}$  is chosen sufficiently large.

Hence, we can find constants  $c_0$  and  $d_0$  with the required property, which completes the proof.

## 4.2 Proof of Corollary 2.6

We place ourselves under the hypothesis of the corollary and let

$$f_{\theta}(x) := h(x) + \beta g_d(\theta) ,$$

where  $g_d(\theta) = \cos(2\pi\theta)^d$ . Let  $C$  and  $E$  be chosen as in (2.10) and (2.11). First of all, we fix some  $\alpha > 1$  and choose  $p \in \mathbb{N}$  such that  $\sup_{x \in C} h'(x) \leq \alpha^{-2/p}$ ,  $\inf_{x \in E} h'(x) > \alpha^{2/p}$ , and in addition  $h'(x) \in (\alpha^{-p}, \alpha^p) \forall x \in \mathbb{T}^1$ . Then  $f$  satisfies  $(\mathcal{A}2)$ – $(\mathcal{A}4)$ .

Let  $\varepsilon := \frac{1}{2}d(h(\mathbb{T}^1 \setminus E), \mathbb{T}^1 \setminus C)$  and suppose  $\beta \in [1 - \varepsilon, 1 + \varepsilon]$ . Define

$$\mathcal{I}_0 := g_d^{-1}([-1 + \varepsilon, -\varepsilon] \cup [\varepsilon, 1 - \varepsilon]) .$$

Then it is easy to see that  $(f_{\theta})_{\theta \in \mathbb{T}^1}$  satisfies  $(\mathcal{A}1)$  and  $(\mathcal{A}7)$ . Further, since

$$|\partial_{\theta} f_{\theta}(x)| = |\beta g'_d(\theta)| = |2\pi\beta d \cdot \cos(2\pi\theta)^{d-1} \cdot \sin(2\pi\theta)| < 4\pi d ,$$

we can choose  $S$  in  $(\mathcal{A}5)$  smaller than  $4\pi d$ .

We check that  $s$  in  $(\mathcal{A}6)$  can be chosen in accordance with (2.8). In order to obtain an estimate  $g'_d$  on  $\mathcal{I}_0$ , we check the endpoints of the connected components and the points where  $g''_d(\theta) = 0$ . Due to the symmetry of  $g_d$ , we can restrict to the interval  $[0, 1/4]$ .

First, assume that  $g_d(\theta) = \varepsilon$ . Then  $\cos(2\pi\theta) = \varepsilon^{1/d}$  and thus  $\sin(2\pi\theta) = \sqrt{1 - \varepsilon^{2/d}}$ . Hence

$$g'_d(\theta) = -2\pi\beta d \cdot \varepsilon^{(d-1)/d} \sqrt{1 - \varepsilon^{2/d}} .$$

Since  $a^y = 1 + \ln(a)y + O(y^2)$  we have

$$\sqrt{d} \sqrt{1 - \varepsilon^{2/d}} = \sqrt{2 \ln \varepsilon + O(1/d)} ,$$

such that for sufficiently large  $d$  there holds

$$|g'_d(\theta)| > \varepsilon \cdot \sqrt{\ln(\varepsilon)} \cdot \sqrt{d} .$$

Secondly, assume that  $g_d(\theta) = 1 - \varepsilon$ . Then  $\cos(2\pi\theta) = (1 - \varepsilon)^{1/d}$  and  $\sin(2\pi\theta) = \sqrt{1 - (1 - \varepsilon)^{2/d}}$ . Thus

$$g'_d(\theta) = -2\pi\beta d \cdot (1 - \varepsilon)^{(d-1)/d} \sqrt{1 - (1 - \varepsilon)^{2/d}}.$$

Similar as above we conclude that for sufficiently large  $d$  there holds

$$|g'_d(\theta)| > (1 - \varepsilon) \cdot \sqrt{\ln(1 - \varepsilon)} \cdot \sqrt{d}.$$

Thirdly, assume that  $g''_d(\theta) = 0$ . In this case  $\sin(2\pi\theta)^2 = 1/d$  and  $\cos(2\pi\theta)^2 = (d - 1)/d$ . Therefore

$$g'_d(\theta) = -2\pi\beta d \left( \frac{d-1}{d} \right)^{(d-1)/2} \frac{1}{\sqrt{d}} = -2\pi\beta\sqrt{d} \left( 1 - \frac{1}{d} \right)^{(d-1)/2},$$

and the last factor is bounded for all  $d$ .

From the above analysis we conclude that there is a constant  $A$ , depending only on  $\varepsilon$ , such that for all sufficiently large  $d$  there holds  $g'_d(\theta) > \sqrt{d}/A$  for all  $\theta \in \mathcal{I}_0$ .

Finally, we let  $\mathcal{I}'_0 := B_{\frac{1}{\sqrt{d}}}(0) \cup B_{\frac{1}{\sqrt{d}}}(\frac{1}{2})$ . Since  $\cos(2\pi\theta) \leq 1 - |\theta|^2$  in a neighbourhood of 0, we obtain that for any  $\theta \in [0, \frac{1}{4}] \setminus \mathcal{I}'_0$  there holds

$$|g_d(\theta)| \leq \left(1 - d^{-2/3}\right)^d \xrightarrow{d \rightarrow \infty} 0.$$

By symmetry, the same estimate holds on all of  $\mathbb{T}^1 \setminus \mathcal{I}'_0$ . Therefore  $\mathcal{I}_0 \subseteq \mathcal{I}'_0$  for large  $d$ .

Similarly, we obtain that for any  $\theta \in \mathbb{T}^1 \setminus \mathcal{I}'_0$  there holds

$$|g'_d(\theta)| \leq 2\pi\beta d \left(1 - d^{-2/3}\right)^{d-1} \xrightarrow{d \rightarrow \infty} 0.$$

Consequently, we can choose  $s'$  in (A8) as a fixed constant, independent of  $d$ , which implies that  $s'/s$  converges to 0 as  $d$  is increased.

This shows that for sufficiently large  $d$  all assumptions of Theorem 2.5 are satisfied, which completes the proof of the corollary.

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